Math 203, Exam #1 **Solutions**
Fall 2008
Please write out the full pledge and sign below.

1. True/False Decide whether each of the following statements is true or false. **Please give a short explanation for your answer.**

a. T **[F]** In some cases, a matrix may be row reduced to two different matrices in reduced echelon form, using different sequences of row operations.

Every matrix has exactly one reduced echelon form.

b. [T] **F** If one row of the echelon form of the augmented matrix for a system of equations is $[0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 2]$, then the system is inconsistent.

This row represents the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 2$, which has no solution.

c. T **[F]** The span of two vectors $\vec{v}$ and $\vec{w}$ is a plane through the origin.

Not necessarily: if $\vec{v}$ is a multiple of $\vec{w}$, then the span of $\vec{v}$ and $\vec{w}$ is a line through the origin.

2. A system of equations has an augmented matrix of the form:

$$\begin{bmatrix} 1 & a & 0 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix}$$

Describe the solutions set.

$x_1 = b - ax_2$, $x_3 = c$, $x_4 = d$, x_2 is free.

in parametric form:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ c \\ d \end{bmatrix} + x_2 \begin{bmatrix} -a \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

3. Suppose you have a consistent system of 2 linear equations with 3 variables x_1, x_2 and x_3. Find all possible forms for the echelon form for the augmented matrix. For each form, determine which variables x_1, x_2 and x_3 are free. [**Note: be sure to only include forms that represent consistent systems**]

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \end{bmatrix} x_3 \text{ free} \qquad \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix} x_2 \text{ free} \qquad \begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} x_2, x_3 \text{ free}$$

$$\begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix} x_1 \text{ free} \qquad \begin{bmatrix} 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare \end{bmatrix} x_1, x_3 \text{ free}$$

$$\begin{bmatrix} 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{bmatrix} x_1, x_2 \text{ free}$$

Technically, there is also

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x_1, x_2, x_3 \text{ free}$$

4. Consider the system of linear equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & & & = & 4 \\ -2x_1 & - & 4x_2 & + & x_3 & - & x_4 & = & -9 \\ 3x_1 & + & 6x_2 & + & 2x_3 & & & = & 1 \end{array}$$

a. Write this system of equations as a matrix equation.

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & -4 & 1 & -1 \\ 3 & 6 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -9 \\ 1 \end{bmatrix}$$

b. Write this system of equations as a vector equation.

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

c. Find the augmented matrix of this system.

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ -2 & -4 & 1 & -1 & -9 \\ 3 & 6 & 2 & 0 & 1 \end{bmatrix}$$

d. Use row operations to find the reduced echelon form of this matrix.

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ -2 & -4 & 1 & -1 & -9 \\ 3 & 6 & 2 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & -1 & 0 & -11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 3 & -1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & -1 & -34 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & 11 \\ 0 & 0 & 0 & 1 & 34 \end{bmatrix}$$

e. Use your answer to part **d** to find the solution set to the system of equations. Please give your answer in parametric form.

$x_1 = -7 - 2x_2$, $x_3 = 11$, $x_4 = 34$, x_2 is free so

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -7 - 2x_2 \\ x_2 \\ 11 \\ 34 \end{bmatrix}$$

In parametric form: $\boxed{\bar{x} = \begin{bmatrix} -7 \\ 0 \\ 11 \\ 34 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$

5. Let $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

a. Compute $2\vec{v} - 3\vec{w}$

$$2\vec{v} - 3\vec{w} = \begin{bmatrix} 5 \\ 1 \\ -3 \end{bmatrix}$$

b. Which of the following vectors are in the span of $\vec{v}$ and $\vec{w}$?

(i) $\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ Yes: this is $2\vec{v}$

(ii) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ No: solving the system with augmented matrix: $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ leads to the matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, which indicates that the system is inconsistent.

Another way to see this is to observe that if the linear combination has 1 as the last coordinate the weight for $\vec{w}$ must be 1 (since the last coordinate of $\vec{v}$ is 0). Then the only way to get a -1 in the second coordinate is for the weight for $\vec{w}$ to be -1. But $-\vec{v} + \vec{w}$ doesn't have first coordinate 1.

(iii) $\begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$ Yes: this is $3\vec{v}+2\vec{w}$. This can be found by inspection or by solving the system with augmented matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 8 \\ 0 & 1 & 2 \end{bmatrix}$

c. For which value(s) of h is the vector $\begin{bmatrix} 1 \\ 2h \\ h \end{bmatrix}$ in the span of $\vec{v}$ and $\vec{w}$?

This requires solving the system with augmented matrix $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2h \\ 0 & 1 & h \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2h \\ 0 & 1 & h \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 2h-2 \\ 0 & 1 & h \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & h \\ 0 & 3 & 2h-2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & h+1 \\ 0 & 1 & h \\ 0 & 0 & -h-2 \end{bmatrix}$$

The last row requires $-h - 2 = 0$, so $\boxed{h = -2}$.

6. Determine which of the following matrices are in echelon form and which are in reduced echelon form, and which are in neither form.

e=echelon form re=reduced echelon form n=neither

When the matrix is in neither form, I've boxed an entry that kept the matrix from being in echelon form. When the matrix is in echelon form, I've boxed an entry that kept the matrix from being in reduced echelon form.

a. $\begin{bmatrix} 1 & \boxed{1} & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ e

b. $\begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ \boxed{1} & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$ n

c. $\begin{bmatrix} 1 & -1 & 0 & \boxed{1} & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ e

d. $\begin{bmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ e (repeat of **c.**)

e. $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ re

f. $\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ re