Math 203, Exam #1 Review Solutions Fall 2008

1. Consider the system of linear equations:

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & 2x_3 & = & 0 \\ x_1 & - & x_2 & - & x_3 & = & 0 \\ -2x_1 & + & x_2 & + & 2x_3 & = & 0 \end{array}$$

a. Find the augmented matrix of this system.

$$\boxed{\begin{bmatrix} 2 & 1 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix}}$$

b. Use row operations to find the reduced echelon form of this matrix.

$$\begin{bmatrix} 2 & 1 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & -2 & 0 \\ -2 & 1 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \to \boxed{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

c. Use your answer to part **b** to find the solution set to the system of equations.
$\boxed{x_3 \text{ is free, } x_1 = x_3,\ x_2 = 0}$. Another way to say this is

$$\boxed{\text{The solutions set is all vectors of the form: } \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} \text{ or } x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

2. Suppose you have a system of 3 linear equations with 4 unknowns.

a. Find all possible forms for the echelon form for the augmented matrix.

There are a bunch:

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{bmatrix} \begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix} \begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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b. Which of these forms represent cases where there is no solution? Exactly one solution? 2 free variables?

The ones that have a black square in the last column have no solution. None of these has exactly one solution. The 4th, 7th, 9th 14th, 16th and 20th have exactly 2 free variables.

3. A system of equations has an augmented matrix of the form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \end{bmatrix}$$

What can you say about the solution set, algebraically and geometrically?

This form tells that story that x_2 is free, x_3 is constant, and $x_1 = ax_2 + b$ for some constants a and b.

Geometrically, the solution set is a line.

4. State the definition of a consistent system of equations.

A system of equations is consistent if there is at least one solution.

5. Let $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

a. Compute $\vec{v} - 2\vec{w}$ $\boxed{\vec{v} - 2\vec{w} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}}$

b. List 3 vectors in the span of $\vec{v}$ and $\vec{w}$.

The span of $\vec{v}$ and $\vec{w}$ is the set of all linear combinations of $\vec{v}$ and $\vec{w}$. Of course there are infinitely many corrects answers. Here's one:

$$\boxed{\begin{bmatrix} 5 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

These correspond to $\vec{v} - 2\vec{w}$, $2\vec{v}$ and $\vec{v} + \vec{w}$ respectively.

c. For which values of h is the vector $\vec{u} = \begin{bmatrix} 2 \\ h \end{bmatrix}$ a linear combination of $\vec{v}$ and $\vec{w}$?

Observe that $\vec{v}$ and $\vec{w}$ are not multiples of one another, so every vector in the plane is a linear combination of $\vec{v}$ and $\vec{w}$. This means $\boxed{\vec{u} \text{ is a linear combination of } \vec{v} \text{ and } \vec{w} \text{ for all values of } h.}$

Alternatively, we can solve the vector equation $x_1\vec{v} + x_2\vec{w} = \vec{u}$. We can solve this using the augmented matrix:

$$\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & h \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 4 & h-2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & .5h+1 \\ 0 & 1 & .25h-.5 \end{bmatrix}$$

We see that $x_1 = .5h + 1$ and $x_2 = .25h - .5$, so for any given value of h, $\vec{u}$ is the linear combination:

$\vec{u} = (.5h+1)\vec{v} + (.25h-.5)\vec{w}$, or $\begin{bmatrix} 2 \\ h \end{bmatrix} = (.5h+1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (.25h-.5)\begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

Since this formula works for any value of h, $\vec{u}$ is a linear combination of $\vec{v}$ and $\vec{w}$ for any value of h.

d. Write a system of equations that is equivalent to $x_1\vec{v} + x_2\vec{w} = \vec{0}$.

$$\begin{array}{rcrcl} x_1 & - & 2x_2 & = & 0 \\ x_1 & + & 2x_2 & = & 0 \end{array}$$

e. Write this system of equations as a matrix equation.

$A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

6. Compute each of the following. If the product is undefined, explain why.

a. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

This is undefined: the number of columns in the matrix must be the same as the number of rows in the column vector.

b. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 2 \end{bmatrix}}$

c. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$

7. Find the solution set to each of the following systems of equations. Give your answer in parametric form.

a. $\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & 2x_3 & = & 0 \\ x_1 & - & x_2 & - & x_3 & = & 0 \end{array}$ **b.** $\begin{array}{rcrcrcl} 2x_1 & + & x_2 & - & 2x_3 & = & 1 \\ x_1 & - & x_2 & - & x_3 & = & 2 \end{array}$

a. $\begin{bmatrix} 2 & 1 & -2 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

x_3 is free, $x_1 = x_3$, and $x_2 = 0$, so $\vec{x} = \begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix}$. In parametric form this is $\vec{x} = x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 & -2 & 1 \\ 1 & -1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 3 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x_3+1 \\ -1 \\ x_3 \end{bmatrix}$. In parametric form this is $\vec{x} = x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

8. Determine whether the following set is linearly independent:

$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 7 \\ 8 \end{bmatrix}$

Using the definition of linearly independent, we need to see if there is a non trivial solution to the equation:

$$x_1\begin{bmatrix}1\\0\\2\\4\end{bmatrix}+x_2\begin{bmatrix}2\\1\\3\\0\end{bmatrix}+x_3\begin{bmatrix}4\\1\\7\\8\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

This is the same as solving the system with augmented matrix:

$$\begin{bmatrix}1&2&4&0\\0&1&1&0\\2&3&7&0\\4&0&8&0\end{bmatrix}.$$

After performing row operations, the matrix is equivalent to:

$$\begin{bmatrix}1&0&2&0\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

Because of the form of this matrix, we can see that x_3 is a free variable, so there is a non-trivial solution. Therefore the vectors are $\boxed{\text{not linearly independent}}$.

9. Determine whether the following set is linearly independent:

$$\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}0\\1\\4\end{bmatrix},\begin{bmatrix}9\\-1\\0\end{bmatrix},\begin{bmatrix}1\\0\\-3\end{bmatrix}$$

Since there are four vectors and only 3 entries in each, the vectors are $\boxed{\text{not linearly independent}}$.

10. Consider the set of vectors $\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}4\\h\end{bmatrix}$

In parts a and b, we are checking for solutions to the system with augmented matrix:

$$\begin{bmatrix}1&4&0\\-2&h&0\end{bmatrix}$$

$$\begin{bmatrix}1&4&0\\-2&h&0\end{bmatrix}\to\begin{bmatrix}1&4&0\\0&h+8&0\end{bmatrix}$$

There will be a free variable (indicating linearly dependent) if $h=-8$, otherwise there is no free variable (indicating linearly independent).

a. Determine a value of h for which this is a linearly independent set.

There are infinitely many correct answers. For example $\boxed{h=0}$

b. Determine a value of h for which this is a linearly dependent set.

$\boxed{h=-8}$.

Note that if $h=-8$, the vectors are $\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}4\\-8\end{bmatrix}$, so the second vector is 4 times the first.

11. Find the matrix that represents the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ with

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}2x_1-x_3\\0\\x_2+x_3\end{bmatrix}$$

$$\begin{bmatrix}2x_1-x_3\\0\\x_2+x_3\end{bmatrix}=x_1\begin{bmatrix}2\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\1\end{bmatrix}+x_3\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}2&0&-1\\0&0&0\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

The matrix is $\boxed{A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}}$

12. Find the matrix that represents the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that extends each vector by a factor of 3.

The matrix is $\boxed{B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$

13. Find the matrix that represents the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that first reflects across the x-axis, then reflects across the line $y = x$. Is there an easier way to describe this transformation?

First we figure out what this transformation does to the vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Reflection across the x-axis doesn't affect $\vec{e}_1$, then reflecting across $y = x$ takes $\vec{e}_1$ to $\vec{e}_2$. So $T(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Reflection across the x-axis takes $\vec{e}_2$ to $-\vec{e}_2$, then reflecting across $y = x$ takes $-\vec{e}_2$ to $-\vec{e}_1$. So $T(\vec{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

Using $T(\vec{e}_1)$ and $T(\vec{e}_2)$ as the columns of the matrix:

The matrix is $\boxed{C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$

14. Is the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+4 \\ y+4 \end{bmatrix}$ a linear transformation? Explain.

No. There are many ways to prove this. In particular, $T(\vec{0}) = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \neq \vec{0}$. But for a linear transformation, $T(\vec{0}) = \vec{0}$.

15. Is the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$ a linear transformation? Explain.

Yes, this is the linear transformation described in problem 13, whose matrix is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$