Math 303, Exam #2 Review Fall 2008

Topics that will be covered on the exam include: Computing and approximating f_x, f_y, f_{xx}, f_{xy}, f_{yy} and $f_{\vec u}$, $\operatorname{grad} f$; Geometric properties of $\operatorname{grad} f(x,y)$ and $\operatorname{grad} g(x,y,z)$; Chain Rule; Computing and using Taylor Polynomials; Using partial derivatives of a function to determine critical points, local maxima, local minima and saddle points; Lagrange multipliers.

In addition to the following problems, you should review each of the worksheets, and homework.

1. Let $\vec v = \vec\imath + \vec\jmath - \vec k$ and $\vec w = 2\vec\imath + \vec\jmath$. Compute $\vec v \times \vec w$.

$\boxed{\vec v \times \vec w = \vec\imath - 2\vec\jmath - \vec k}$

2. Find a formula for the plane containing the points (0,1,1), (1,2,-1) and (-1,3,0).

Let $P = (0,1,1)$, $Q = (1,2,-1)$ and $R = (-1,3,0)$, and set $\vec v = \vec{PQ} = \vec\imath + \vec\jmath - 2\vec k$ and $\vec w = \vec{PR} = -\vec\imath + 2\vec\jmath - \vec k$. Then $\vec v \times \vec w$ is normal to this plane, and $\vec v \times \vec w = 3\vec\imath + 3\vec\jmath + 3\vec k$. This, together with any of the points will give us the equation of the plane:

$\boxed{3(x-0)+3(y-1)+3(z-1)=0}$ or $\boxed{3x+3y+3z=6}$

3. Let $f(x,y) = 3x^2y - 4xy^2$, and let $\vec v = \vec\imath - 2\vec\jmath$. Compute f_x, f_y, f_{xx}, f_{xy}, f_{yy}, $f_{\vec v}$, and $\operatorname{grad} f$. Find the critical points of f. Illustrate $\operatorname{grad} f(1,2)$ on a graph of the function f.

$\boxed{f_x = 6xy - 4y^2}$ $\boxed{f_y = 3x^2 - 8xy}$ $\boxed{f_{xx} = 6y}$ $\boxed{f_{xy} = 6x - 8y}$ $\boxed{f_{yy} = -8x}$

$\boxed{\operatorname{grad} f = (6xy - 4y^2)\vec\imath + (3x^2 - 8xy)\vec\jmath}$

$\operatorname{grad} f(1,2) = -4\vec\imath - 13\vec\jmath$, and $\vec u = \frac{1}{\sqrt5}(\vec\imath - 2\vec\jmath)$ is the unit vector in the same direction as $\vec v$.

$f_{\vec v} = \operatorname{grad} f \cdot \vec u = \frac{1}{\sqrt5}(4\vec\imath - 13\vec\jmath)\cdot(\vec\imath - 2\vec\jmath) = \frac{1}{\sqrt5}(4+26)\boxed{= 6\sqrt5}$

4. Let $g(x,y,z) = \dfrac{x+y}{z^2+1}$. Compute $\operatorname{grad} g$. Compute $\operatorname{grad} g(1,1,1)$. Describe $\operatorname{grad} g$ geometrically.

$\boxed{\operatorname{grad} g = \left(\dfrac{1}{z^2+1}\right)\vec\imath + \left(\dfrac{1}{z^2+1}\right)\vec\jmath - \left(\dfrac{2z(x+y)}{(z^2+1)^2}\right)\vec k}$ $\boxed{\operatorname{grad} g(1,1,1) = \dfrac{1}{2}\vec\imath + \dfrac{1}{2}\vec\jmath - \vec k}$

$\operatorname{grad} g(1,1,1)$ is normal to the level surface of $g(x,y,z)$ containing the point (1,1,1) at the point (1,1,1). It points in the direction of maximal increase in f.

5. $f(x,y) = x^2 - y^2$, $x = uv^2 + v$, and $y = uv^2 - v$. Use the chain rule to compute f_u and f_v.

$f_u = \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 2xv^2 - 2yv^2 = 2v^2(x-y) = 2v^2((uv^2+v) - uv^2 - v) = 2v^2(2v)$

$\boxed{f_u = 4v^3}$

$f_v = \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = 2x(2uv+1) - 2y(2uv-1) = 2(uv^2+v)(2uv+1) - 2(uv^2-v)(2uv-1) = 4u^2v^3 + 4uv^2 + 2uv^2 + v - (4u^2v^3 - 4uv^2 - 2uv^2 + v)$ $\boxed{f_v = 12uv^2}$

6. Let $f(x,y) = xy\sin(x+y)$. Find the first and second Taylor polynomials for $f(x,y)$ near $(\frac{\pi}{4}, \frac{\pi}{4})$.

$f(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{\pi}{4}\cdot\frac{\pi}{4}\cdot\sin(\frac{\pi}{2}) = \frac{\pi^2}{16}$

$f_x = y\sin(x+y) + xy\cos(x+y)$, $f_x(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{\pi}{4}$

$f_y = x\sin(x+y) + xy\cos(x+y)$, $f_y(\frac{\pi}{4}, \frac{\pi}{4}) = \frac{\pi}{4}$

$f_{xx} = 2y\cos(x+y) - xy\sin(x+y)$, $f_{xx}(\frac{\pi}{4}, \frac{\pi}{4}) = -\frac{\pi^2}{16}$

$f_{yy} = 2x\cos(x+y) - xy\sin(x+y)$, $f_{yy}(\frac{\pi}{4}, \frac{\pi}{4}) = -\frac{\pi^2}{16}$

$f_{xy} = \sin(x+y) + y\cos(x+y) + x\cos(x+y) - xy\sin(x+y)$, $f_{xy}(\frac{\pi}{4}, \frac{\pi}{4}) = 1 - \frac{\pi^2}{16}$

$$\boxed{T_1(x,y) = \frac{\pi^2}{16} + \frac{\pi}{4}\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}\left(y - \frac{\pi}{4}\right)}$$

$$\boxed{T_2(x,y) = \frac{\pi^2}{16} + \frac{\pi}{4}\left(x - \frac{\pi}{4}\right) + \frac{\pi}{4}\left(y - \frac{\pi}{4}\right) - \frac{\pi^2}{32}\left(x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{32}\left(y - \frac{\pi}{4}\right)^2 + \left(1 - \frac{\pi^2}{16}\right)\left(x - \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right)}$$

7. Find the maximum and minimum of the function $f(x,y) = x(y^2 - 1)$ subject to the constraint $x^2 + y^2 \leq 4$.

$f_x = y^2 - 1$, and $f_y = 2xy$

$f_x = 0$ if $y = \pm 1$

$f_y = 0$ for $x = 0$ or $y = 0$. If $y = 0$, $f_x \neq 0$

For $f_x = f_y = 0$, we must have $x = 0$ and $y = \pm 1$, so the critical points are $(0, \pm 1)$.

Now we apply Lagrange multipliers to the function $f(x,y) = x(y^2 - 1)$ with $g(x,y) = x^2 + y^2 = 4$

$\operatorname{grad} f = (y^2 - 1)\vec{\imath} + 2xy\vec{\jmath}$, and $\operatorname{grad} g = 2x\vec{\imath} + 2y\vec{\jmath}$, so if $\operatorname{grad} f = \lambda \operatorname{grad} g$, then:

$$\begin{aligned} y^2 - 1 &= \lambda 2x \\ 2xy &= \lambda 2y \end{aligned}$$

Dividing the first equation by $2x$ and the second equation by $2y$ gives $\frac{y^2-1}{2x} = \lambda = x$. Solving for y^2 gives $y^2 = 2x^2 + 1$. This together with the constraint equation $x^2 + y^2 = 4$ gives us the points $(\pm 1, \pm\sqrt{3})$.

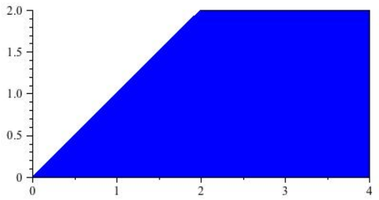
Since we divided by x and y we have to consider the possibility that $x = 0$ or $y = 0$. Using the constraint equation again gives us the points $(0, \pm 2)$ and $(\pm 2, 0)$.

All together the candidates for max and min are $(0, \pm 1)$, $(\pm 1, \pm\sqrt{3})$, $(0, \pm 2)$, $(\pm 2, 0)$. Evaluating $f(x,y)$ at each point yields

f has a maximum of 2 at the points $(-2, 0)$ and $(1, \pm\sqrt{3})$, and a minimum of -2 at the points (2,0) and $(-1, \pm\sqrt{3})$.

8. Sketch the region determined by $y \leq x \leq 4$, $0 \leq y \leq 2$.

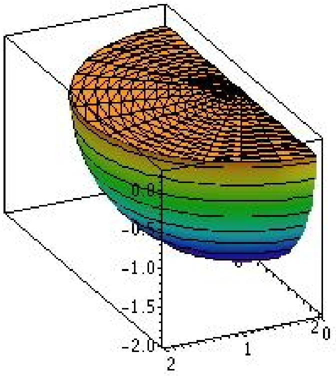
Draw the lines $x = y$, $x = 4$, $y = 0$ and $y = 2$, and take the region enclosed:



9. Sketch and/or describe the solid determined by $-\sqrt{4-x^2-y^2} \le z \le 0$, $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, $0 \le x \le 2$.

The limits $-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$, $0 \le x \le 2$ describe the positive x half of a circle of radius 2 centered at the origin in the xy-plane. The limits $-\sqrt{4-x^2-y^2} \le z \le 0$ describe the negative z half of a sphere of radius 2 centered at the origin. All together these limits describe

$$\boxed{\text{The positive } x \text{ and negative } z \text{ quarter of a sphere with radius 2 centered at the origin.}}$$



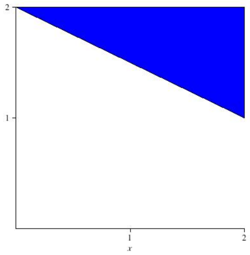
10. Let $f(x,y) = x^2 + y^2$, and let R be the disk with radius 4 in the xy-plane. Set up the iterated integral for $\int_R f\, dA$ using Cartesian coordinates. Set up the iterated integral for $\int_R f\, dA$ using polar coordinates.

In Cartesian coordinates: $\displaystyle\int_R f\, dA = \boxed{\int_{y=-4}^{4} \int_{x=-\sqrt{16-y^2}}^{x=\sqrt{16-y^2}} x^2 + y^2 \; dxdy}$

In polar coordinates: $x^2 + y^2 = r^2$, and $\displaystyle\int_R f\, dA = \int_{r=0}^{4} \int_{\theta=0}^{2\pi} r^2 \cdot r \; d\theta dr = \boxed{\int_0^4 \int_0^{2\pi} r^3 \; d\theta dr}$

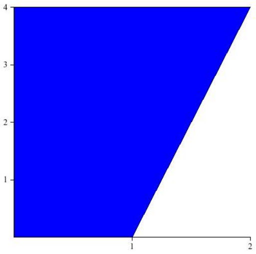
This illustrates the advantage of polar coordinates!

11. Set up the integral of a function $f(x,y)$ over each of the following regions:



The triangle is bounded by the lines
$y = 2$, $x = 2$ and $y = 2 - \frac{1}{2}x$

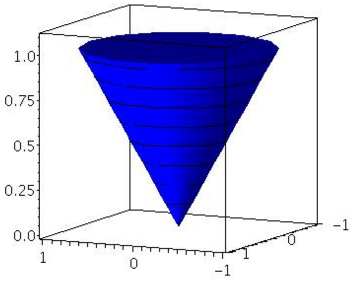
$$\boxed{\int_0^2 \int_{2-\frac{1}{2}x}^{2} f(x,y)\, dydx}$$



The quadrilateral is bounded by the lines
$y = 0$, $y = 4$, $x = 0$ and $y = 4x - 4$.
Rewrite the line $y = 4x - 4$ as $x = \frac{1}{4}y + 1$, so
$0 \le x \le \frac{1}{4}y + 1$, and $0 \le y \le 4$

$$\boxed{\int_0^4 \int_0^{\frac{1}{4}y+1} f(x,y)\, dxdy}$$

12. Set up the integral of a function $f(x,y,z)$ over each of the following regions:

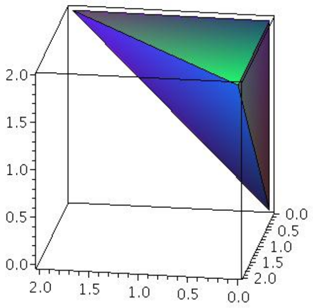


The surface of the cone is described by the function $z = \sqrt{x^2+y^2}$, so we can use
$\sqrt{x^2+y^2} \le z \le 1$.
But we want to restrict to the x and y values in the unit disk:
$-1 \le x \le 1$, $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$.

$$\boxed{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dzdydx}$$

Note that we could do this one using cylindrical coordinates, because $x^2+y^2=r^2$, so the surface of the cone would be $z=r$, and the limits for z would be $r \le z \le 1$, and the limits for r and θ would be $0 \le r \le 1$, $0 \le \theta \le 2\pi$. We would also have to translate $f(x,y,z)$ into cylindrical and compute:

$$\boxed{\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{1} f(r,\theta,z)\cdot r\,dzdrd\theta}$$



The solid is bounded by the planes $z=2$, $x=0$, $y=0$, and a plane containing the points (0,0,0), (2,0,2) and (0,2,2). This plane is $z=x+y$. Then the limits for z are $x+y \le z \le 2$.
But now we want to restrict x and y to the triangle with vertices (0,0), (2,0) and (0,2). This triangle is bounded by the lines $x=0$, $y=0$ and $y=2-x$. So we use
$0 \le x \le 2$, $0 \le y \le 2-x$.

$$\boxed{\int_{0}^{2}\int_{0}^{2-x}\int_{x+y}^{2} f(x,y,z)\,dzdydx)}$$