Math 203, Exam #2 **Name:**________________________________
Fall 2008
Please write out the full pledge and sign below.

Please show all your work and explain your answers on each problem.

1. True/False Decide whether each of the following statements is true or false. **Please give a short explanation for your answer.**

a. **T** **[F]** The map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+1 \\ y \end{bmatrix}$ is a linear transformation.

b. **[T]** **F** If A is an $n \times n$ matrix and the columns of A span $\mathbb{R}^n$, then the columns of A are linearly independent.

c. **[T]** **F** If A is an $n \times n$ matrix and the equation $A\vec{x} = \vec{0}$ has a nontrivial solution, then A has fewer than n pivot positions.

2. Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

a. Give and example of a vector $\vec{v}_3$ so that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

b. Give and example of a vector $\vec{v}_3$ so that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

$$\boxed{\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}}$$

3. Find the matrix for the transformation of $\mathbb{R}^2$ that first rotates counterclockwise by $\frac{\pi}{4}$ radians about the origin, then reflects across the line $y = x$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}$$

4. Find the 3× 3 matrix that produces the following transformation of $\mathbb{R}^2$ using homogeneous coordinates: First reflect across the x-axis, then shift up by 2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

5. Suppose a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 0 \end{bmatrix}$.

Express your answers to the following in simplest form.

a. Find the null space of T.

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_3$, $x_2 = -4x_3$, x_3 is free. The null space is the set of vectors of the form $\bar{x} = x_3 \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$

b. Find the image of T.

The image of T is the set of linear combinations of the columns of A. Since the third column of A is in the span of the first two columns of A, the image is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. More simply, this is the set of vectors of the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$, where $a, b \in \mathbb{R}$.

6. Solve the following system of equations by using the inverse of an appropriate matrix.

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ 3x_1 - x_2 &= 2 \end{aligned}$$

$A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$. The solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}}$$

7. Let A be an $n \times n$ matrix. Give 3 statements that are equivalent to the statement: "A is an invertible matrix."

These can be found in the Invertible Matrix Theorem.

8. Which of the following matrices are invertible? Explain your answer using as few calculations as possible.

a. $\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 4 \\ 3 & 3 & 4 \end{bmatrix}$ Not invertible: row 3 is row 1 plus row 2.

b. $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$ Invertible: echelon form has 3 pivots.

9. Let $B = \begin{bmatrix} 2 & 1 & -1 \\ 4 & -4 & 8 \\ -6 & 3 & -8 \end{bmatrix}$

a. Find an LU factorization of B

$$\begin{bmatrix} 2 & 1 & -1 \\ 4 & -4 & 8 \\ -6 & 3 & -8 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -1 \\ 0 & -6 & 10 \\ 0 & 6 & -11 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & -1 \\ 0 & -6 & 10 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

b. Use your factorization to solve the matrix equation $B\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$

Solve: $L\vec{y} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$, $U\vec{x} = \vec{y}$.

$$L\vec{y} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} : \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 1 & 0 & 2 \\ -3 & -1 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & -1 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$U\vec{x} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} : \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & -6 & 10 & -4 \\ 0 & 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & -6 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \quad \boxed{\vec{x} = \begin{bmatrix} \frac{3}{2} \\ -1 \\ -1 \end{bmatrix}}$$

10. Determine whether the vector $\vec{b} = \begin{bmatrix} 8 \\ 3 \\ -4 \end{bmatrix}$ is in the subspace of $\mathbb{R}^3$ spanned by

$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 8 \\ 1 & 0 & 3 \\ 2 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 8 \\ 2 & -1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix}$$

This is consistent $\boxed{\text{yes}}$

11. Give an example of a basis for $\mathbb{R}^3$ other than the standard basis.

We just need 3 linearly independent vectors in $\mathbb{R}^3$. For example:

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}}$$

12. Let $A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ 4 & 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 9 & 0 \\ 1 & -8 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 4 \\ 4 & 2 & 8 \end{bmatrix}$

Compute the following, or explain why the expression is undefined.

a. AB

Not defined: (number of columns of A) $\neq$ (number of rows of B)

b. BA

$$BA = \begin{bmatrix} 1 & 9 & 0 \\ 1 & -8 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 4 \\ 4 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 36 \\ 6 & -8 & -20 \end{bmatrix}$$

c. The determinant of C.

$$2\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 2(8-8) + (4-4) = 0$$

d. C^{-1}

Doesn't exist: $\det(C) = 0$

e. A^{-1}

$$\begin{bmatrix} -2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 3 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 4 & 0 & 1 & 0 \\ 0 & 2 & 8 & 1 & 2 & 0 \\ 0 & -4 & -13 & 0 & -4 & 1 \end{bmatrix} \to$$

$$\left[\begin{array}{cccccc} 1 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & -4 & -13 & 0 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 3 & 2 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 4 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -\frac{13}{6} & 1 & -\frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \left[\begin{array}{ccc} -\frac{1}{2} & 0 & 0 \\ -\frac{13}{6} & 1 & -\frac{4}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \end{array}\right]$$