Math 203, Exam #3 Review Fall 2009

1. Suppose $A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, Where B, C and D are 3×3 matrices. Find a formula for A^{-1}. Are there any conditions on B, C and D?

2. Suppose a transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ is represented by the matrix $A = \begin{bmatrix} 1 & 2 & -3 & 3 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

- **a.** Find the null space of T.
- **b.** Find the image of T.

3. Suppose a transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ is represented by the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

- **a.** Find the null space of T.
- **b.** Find the image of T.

4. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & -2 \\ 4 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 2 \\ 0 & -1 \\ 2 & 3 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 3 \\ -2 & 3 \\ 0 & 1 \end{bmatrix}$

5. Which of the following matrices are invertible? Use as few calculations as possible.
$\begin{bmatrix} 1 & 3 & 3 \\ -2 & 4 & 4 \\ 0 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ 5 & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 3 \\ 0 & 2 & 8 \\ 0 & 1 & 4 \end{bmatrix}$

6. Draw an example of a subspace of $\mathbb{R}^2$.

7. Draw an example of a subset of $\mathbb{R}^2$ that is not a subspace.
$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$.

8. Give an example of a basis for $\mathbb{R}^2$ other than the standard basis. Do the same for $\mathbb{R}^3$. Give an example of a pair of vectors in $\mathbb{R}^2$ that do not form a basis for $\mathbb{R}^2$.

9. Find the determinant of the matrices $A = \begin{bmatrix} 4 & 3 \\ -4 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 4 \\ -1 & 0 & 0 \end{bmatrix}$.

10. Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 12$, $\det \begin{pmatrix} b & c \\ h & i \end{pmatrix} = 5$, $\det \begin{pmatrix} a & c \\ g & i \end{pmatrix} = 9$ and $\det \begin{pmatrix} a & b \\ g & h \end{pmatrix} = 3$. Compute the determinant of each of the following.

- **a.** $\begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$ $\det \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} = -det \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix} = det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \boxed{12}$
- **b.** $\begin{pmatrix} a & b & c \\ 2a & 2b & 2c \\ d & e & f \end{pmatrix}$ The second row is a multiple of the first row $\boxed{0}$

c. $\begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{pmatrix}$ $\det\begin{pmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{pmatrix} = 2\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \boxed{24}$

d. $\begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix}$ $\det\begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix} = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \boxed{12}$

e. $\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix}$

Expand down the first column: $\det\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix} = 3\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \boxed{36}$

f. $\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & a & b & c \\ 1 & d & e & f \\ 0 & g & h & i \end{pmatrix}$

Expand down the first column:

$$\begin{aligned} \det\begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix} &= 3\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} + \det\begin{pmatrix} 2 & 1 & 0 \\ a & b & c \\ g & h & i \end{pmatrix} \\ &= 36 + 2\det\begin{pmatrix} b & c \\ h & i \end{pmatrix} - \det\begin{pmatrix} a & c \\ g & i \end{pmatrix} \\ &= 36 + 2(5) - 9 = \boxed{37} \end{aligned}$$

11. Determine which of the following are subspaces of $\mathbb{P}_6$.

a. $S = \{at^2 \mid a \in \mathbb{R}\}$

This is subspace of $\mathbb{P}_6$. [You can check that $0 \in S$, and the sums of elements of S are in S and that scaler multiples of elements in S are in S.]

b. $T = \{a + t^2 \mid a \in \mathbb{R}\}$

This is not. For one thing, $0 \notin T$. For another, $t^2 \in T$ but $2t^2 \notin T$.

c. $U = \{p(t) \in \mathbb{P}_6 \mid p(0) = 0\}$

This is subspace of $\mathbb{P}_6$. [You can check that $0 \in U$, and the sums of elements of U are in U and that scaler multiples of elements in U are in U.]

d. The set, V, of polynomials of even degree at most 6.

This is subspace of $\mathbb{P}_6$. [You can check that $0 \in V$, and the sums of elements of V are in V and that scaler multiples of elements in S are in V.]

12. Find a basis for $\mathbb{P}_3$ other than the standard basis. Find $[2 + 3t + t^3]_{\mathcal{B}}$ with respect to this basis.

One example is $\mathcal{B} = \{1, 1+t, 1+t+t^2, 1+t+t^2+t^3\}$, but of course there are lots of other examples.

To find $[2+3t+t^3]_{\mathcal{B}}$, we need to find x_1, x_2, x_3, x_4 so that

$$x_1(1) + x_2(1+t) + x_3(1+t+t^2) + x_4(1+t+t^2+t^3) = 2+3t+t^3.$$

We can do this directly, or using the matrix $P_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, and solving $P_{\mathcal{B}}\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$. Using either the augmented matrix or the matrix inverse, gives $\boxed{\vec{x} = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 1 \end{bmatrix}}$. You can check this by computing:

$$-1(1) + 3(1+t) - 1(1+t+t^2) + 1(1+t+t^2+t^3) = 2+3t+t^3 \checkmark$$

13. Find a basis for the subpace of $\mathbb{R}^4$ consisting of vectors of the form $\begin{bmatrix} c-2d \\ 2d \\ c \\ d \end{bmatrix}$, $c, d \in \mathbb{R}$.

$$\begin{bmatrix} c-2d \\ 2d \\ c \\ d \end{bmatrix} = c\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad \boxed{\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

14. Let $A = \begin{pmatrix} 4 & 6 & 14 & 0 & 2 \\ 2 & 3 & 10 & -1 & 0 \\ 6 & 9 & 18 & 1 & 10 \\ 2 & 3 & 4 & 1 & 8 \end{pmatrix}$. Find bases for and compute the dimension of each of the following spaces:

a. Null(A)

b. Col(A)

c. Row(A)

First row-reduce A, to get $B = \begin{pmatrix} 1 & \frac{3}{2} & 0 & \frac{7}{6} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Null(A) has basis $= \left\{ \begin{bmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{bmatrix} \right\}$, so $\dim \text{Null}(A) = 2$

Col(A) has basis $= \left\{ \begin{bmatrix} 4 \\ 2 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 14 \\ 10 \\ 18 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 10 \\ 8 \end{bmatrix} \right\}$, so $\dim \text{Null}(A) = 3$

Row(A) has basis $= \{(1, 3/2, 0, 7/6, 0), (0, 0, 1, -1/3, 0), (0, 0, 0, 0, 1)\}$, so $\dim \text{Row}(A) = 3$

15. Find a matrix M so that $\text{Col}(M) = \left\{ \begin{bmatrix} a+b \\ b+c-d \\ a+2b-c \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$

$$\boxed{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 0 \end{bmatrix}}$$

16. Determine which of the following are bases for $\mathbb{R}^3$. Give a short explanation for your answer.

a. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ No: this set doesn't span.

b. $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ Yes: You can check that these are three linearly independent vectors by row reducing, or by observing that the first two are clearly independent, and both have a 0 in the second position but the third does not. Any three linearly independent vectors in $\mathbb{R}^3$ form a basis.

c. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ No: This set contains $\vec{0}$, so it is not linearly independent.

d. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ No: This set contains more than three vectors in $\mathbb{R}^3$, so it is not linearly independent.

17. Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \right\}$.

a. Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

$\mathcal{B}$ is a basis for $\mathbb{R}^3$, since the matrix $A = \begin{bmatrix} 2 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix}$ is row equivalent to the identity.

b. Find $[\vec{x}]_{\mathcal{B}}$, where $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

This means solving the equation: $A\,[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Since A is invertible, this equation has unique solution: $[\vec{x}]_{\mathcal{B}} = A^{-1} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 1/2 \end{bmatrix}$

c. Find $\vec{x}$, where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$.

$\vec{x} = A \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$

18. Let A be an $n \times n$ matrix. Give three of the new statements that are equivalent to the statement A is an invertible matrix.

These are listed on page 267 of the text.

19. A particular strain of flu is going around campus. There are students who have never had the flu, students who have the flu now, and students who are recovered from the flu and now immune. Each week, 5% of the students who never had the flu will get the flu and 20% of the student with flu will recover and become immune.

a. Set up the stochastic matrix for this situation.

$A = \begin{bmatrix} .95 & 0 & 0 \\ .05 & 0 & 0 \\ 0 & .2 & 1 \end{bmatrix}$

b. If we start with 80% of the students never having had the flu, and 20% having the flu now, what will the situation be after 10 weeks assuming the trend continues?

Initial state is represented by $\vec{x} = \begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix}$

After 10 weeks, the state vector will be $A^{10}\vec{x} = \begin{bmatrix} .4790 \\ .1525 \\ .3685 \end{bmatrix}$

47.90% never had flu, 15.25% have flu and 36.85% immune.

c. Is there a steady state for this situation? If so, find it.

Yes. $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a steady state.

20. Each month, 10% of the employed people lose their job, and half of the unemployed people get a job.

a. Set up the stochastic matrix for this situation.

$B = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix}$

b. Starting with 7% unemployment, what will the situation be in one year, assuming this trend continues?

$\vec{x} = \begin{bmatrix} .93 \\ .07 \end{bmatrix}$, $B^{12}\vec{x} = \begin{bmatrix} 0.8333 \\ 0.1667 \end{bmatrix}$ 16.67% unemployment