

The exam will cover §17.1-17.5 and §18.1-18.3. In addition to these problems, you will want to review worksheets 12-16 (Note: Worksheet 12 was mislabeled as Worksheet 11. It is the one about parametrizing curves). For the exam, you will be responsible for following tasks using Maple: plotting a parametrized curve, plotting a vector field, superimposing a curve on a vector field.

1. Parametrize each of the following curves.

a. The left half of the circle of radius 3 centered at the origin, starting at (0,-3).

Here are two possibilities:

$$\boxed{x = -3 \sin t, y = -3 \cos t, 0 \leq t \leq \pi.} \quad \boxed{x = 3 \sin t, y = 3 \cos t, \pi \leq t \leq 2\pi.}$$

b. The line from the point (1,1) to the point (-1,2).

$$\boxed{x = 1 - 2t, y = 1 + t, 0 \leq t \leq 1.}$$

c. The line from the point (1,0,1) to the point (-1,2,0).

$$\boxed{x = 1 - 2t, y = 2t, z = 1 - t, 0 \leq t \leq 1.}$$

2. Do the lines

$$\vec{r}_1 = t\vec{i} + (t-1)\vec{j} + (2t-1)\vec{k} \quad (1)$$

$$\vec{r}_2 = 2t\vec{i} + (t+1)\vec{j} + (3t+1)\vec{k} \quad (2)$$

intersect? If so, where?

This happens if there are times t_1 and t_2 so that $\vec{r}_1(t_1) = \vec{r}_2(t_2)$. Then we have to solve the system of equations:

$$\begin{aligned} t_1 &= 2t_2 \\ t_1 - 1 &= t_2 + 1 \\ 2t_1 - 1 &= 3t_2 + 1 \end{aligned}$$

This is equivalent to:

$$\begin{aligned} t_1 - 2t_2 &= 0 \\ t_1 - t_2 &= 2 \\ 2t_1 - 3t_2 &= 2 \end{aligned}$$

We can solve this system by: (i). substituting $t_1 = 2t_2$ in the second equation, (ii) solving for t_2 , to get $t_2 = 2$, (iii) plugging back into $t_1 = 2t_2$ to find $t_1 = 4$, (iv) checking that these values for t_1 and t_2 work in all three equations.

Alternatively, using Linear Algebra you can solve by row-reducing the augmented matrix

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & 2 \\ 2 & -3 & 2 \end{bmatrix} \text{ to get } \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \text{ so that } t_1 = 4 \text{ and } t_2 = 2.$$

Either way,

$$\vec{r}_1(t_1) = \vec{r}_1(4) = 4\vec{i} + 3\vec{j} + 7\vec{k}, \text{ so } \boxed{\text{the two lines intersect at the point } (4, 3, 7).}$$

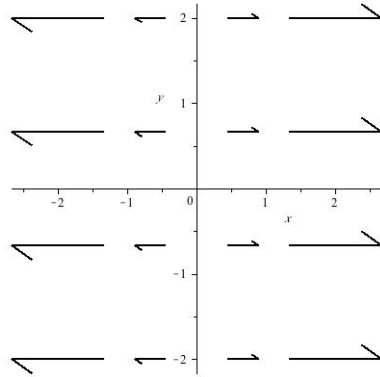
3. Find the velocity vector for the curve parametrized by $x = t^2 - t$, $y = \sin(2t)$. Determine when the object is moving parallel to the x -axis and when it is moving parallel to the y -axis.

$$\vec{v} = (2t - 1)\vec{i} + 2\cos(2t)\vec{j}$$

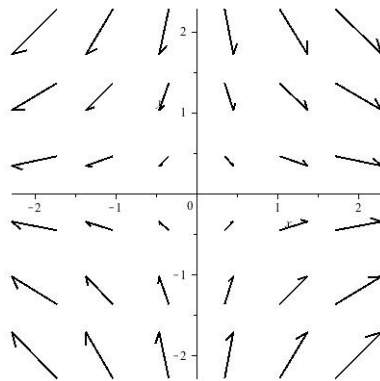
The object is moving parallel to the x -axis when the coefficient of \vec{j} is 0, so when $2\cos(2t) =$

0. This means $2t = \frac{\pi}{2} + k\pi$, or $t = \frac{\pi}{4} + \frac{k\pi}{2}, k \in \mathbb{Z}$

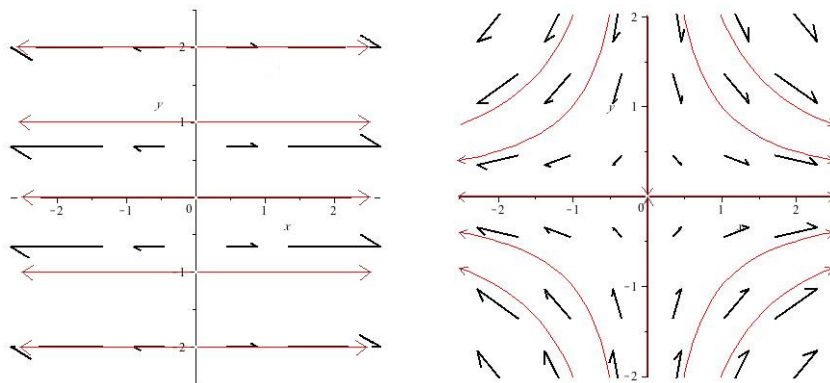
4. Sketch the vector field $\vec{F} = x\vec{i}$.



5. Sketch the vector field $\vec{F} = x\vec{i} - y\vec{j}$.



6. Sketch the flow of each of the vector fields above.



7. Parametrize each of the following surfaces:

a. The surface of a sphere of radius 3.

$$\text{Use spherical coordinates: } \rho = 3, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$$

b. The surface of a cone with radius 4 and height 5, centered at the origin.

In general, the surface of a cone is given by

$x = ar \cos(t), y = ar \sin(t), z = r$, where $0 \leq t \leq 2\pi, 0 \leq r \leq \text{height of cone}$, and a is a constant.

If the height of the cone is 5, then $0 \leq r \leq 5$. We need to adjust a so that the radius of the cone is 4:

$$x = \frac{4r}{5} \cos(t), y = \frac{4r}{5} \sin(t), z = r, \text{ where } 0 \leq t \leq 2\pi, 0 \leq r \leq 5$$

This makes a cone with vertex at the origin and with base at height $z = 5$. If we want the base in the xy -plane use:

$$x = \frac{4r}{5} \cos(t), y = \frac{4r}{5} \sin(t), z = 5 - r, \text{ where } 0 \leq t \leq 2\pi, 0 \leq r \leq 5$$

8. Let $\vec{F}(x, y) = x\vec{i} - y\vec{j}$.

a. Use the graph of \vec{F} to determine which of the following parametrized curves could be flow lines for this vector field.

(i) $\vec{r}_1(t) = \cos(t)\vec{i} - \sin(t)\vec{j}$

(ii) $\vec{r}_2(t) = e^t\vec{i} - e^t\vec{j}$

(iii) $\vec{r}_4(t) = e^t\vec{i} + e^{-t}\vec{j}$

(iv) $\vec{r}_5(t) = 2e^t\vec{i} + e^{-t}\vec{j}$

(v) $\vec{r}_1(t) = \cos(t)\vec{i} + \sin(t)\vec{j}$

(vi) $\vec{r}_5(t) = e^{2t}\vec{i} + e^{-2t}\vec{j}$

(vii) $\vec{r}_3(t) = e^t\vec{i} - e^{-t}\vec{j}$

This is the same vector field graphed in problem b. From the graphs we can rule out (i) and (v) (which are circles), and (ii) which is a line with slope 1. The rest have graphs that look like flow lines.

b. Write down the differential equations associated to the vector field. For each of the the curves that could be flow lines, show that the curve satisfies the differential equations.

The differential equations for this vector field are: $\frac{dx}{dt} = x, \frac{dy}{dt} = -y$.

Testing just (iii), (iv), (vi) and (vii) :

(iii) $\frac{dx}{dt} = e^t = x, \frac{dy}{dt} = -e^{-t} = -y$. Yes.

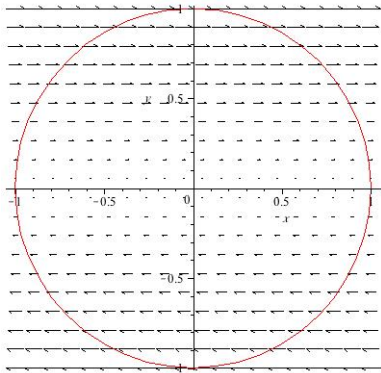
(iv) $\frac{dx}{dt} = 2e^t = 2x, \frac{dy}{dt} = -e^t = -y$. Yes.

(vi) $\frac{dx}{dt} = 2e^{2t} = 2x \neq x$ No.

(vii) $\frac{dx}{dt} = e^t = x, \frac{dy}{dt} = \frac{d}{dt}(-e^{-t}) = e^{-t} = -y$ Yes.

9. Let $\vec{F} = y\vec{i}$. Determine whether each of the following is positive, negative or zero.

a. $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve $\vec{r}(t) = \cos t\vec{i} + \sin t\vec{j}, 0 \leq t \leq 2\pi$.



The circle is traced out counterclockwise, so it is always running against the flow of the vector field. negative

b. $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve $\vec{r}(t) = 2\vec{i} + te^{2t}\vec{j}$, $0 \leq t \leq 3$.

Without any calculations, we can see that this parametrization always runs in a vertical direction, so it is always perpendicular to the flow of the vector field. 0

10. Let $\vec{F} = x\vec{i} + (2x + y)\vec{j} + (3y - z)\vec{k}$, and let C be the circle of radius 2 in the xy -plane traversed clockwise. Compute $\int_C \vec{F} \cdot d\vec{r}$.

$$\vec{r}(t) = 2 \sin(t)\vec{i} + 2 \cos(t)\vec{j}, \quad 0 \leq t \leq 2\pi,$$

$$\vec{r}'(t) = 2 \cos(t)\vec{i} - 2 \sin(t)\vec{j},$$

$$\vec{F}(\vec{r}(t)) = 2 \sin(t)\vec{i} + (4 \sin(t) + 2 \cos(t))\vec{j} + (6 \cos(t))\vec{k}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = 4 \sin(t) \cos(t) - 8 \sin^2(t) - 4 \sin(t) \cos(t) = -8 \sin^2(t).$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -8 \sin^2(t) dt = \boxed{-8\pi}$$

11. Let $\vec{F} = xy\vec{j}$. Is \vec{F} path-independent? Why or why not?

$$\text{No: } \frac{\partial F_1}{\partial y} = 0 \neq y = \frac{\partial F_2}{\partial x}$$

12. Let $\vec{F} = x^2\vec{i} + e^y\vec{j}$. Is \vec{F} path-independent? Why or why not? Let C be the top half of a circle of radius 4 traversed clockwise. Compute $\int_C \vec{F} \cdot d\vec{r}$.

Yes: $\vec{F} = \text{grad } f$, where $f(x, y) = \frac{1}{3}x^3 + e^y$, so we can use the Fundamental Theorem of Line Integrals. C starts at $(-4, 0)$ and ends at $(4, 0)$, so

$$\int_C \vec{F} \cdot d\vec{r} = f(4, 0) - f(-4, 0) = \left(\frac{1}{3}(4^3) + 1\right) - \left(\frac{1}{3}(-4^3) + 1\right) = \boxed{\frac{128}{3}}.$$