Math 203, Final **Name:**______________________________
Fall 2008
Please write out the full pledge and sign below.

Please show all your work and explain your answers on each problem.

1. True/False Decide whether each of the following statements is true or false. **Please give a short explanation for your answer.**

a. T F If A is an $n \times n$ matrix and the columns of A are linearly independent, then A is invertible.

b. T F If the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set, then $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

c. T F An inconsistent system of equations has more than one solution.

d. T F $2\vec{v}_1$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

e. T F If A is a matrix, then the rank of A is the number of pivots of A.

f. T F If 0 is and eigenvector of A, then A is not invertible.

g. T F The determinant of A is the product of the diagonal entries of A.

2. Suppose $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 5$. Compute the determinant of each of the following.

a. $\begin{pmatrix} c & d \\ 4a & 4b \end{pmatrix}$

b. $\begin{pmatrix} a-2c & b-2d \\ c & d \end{pmatrix}$

c. $\begin{pmatrix} 0 & 3 & 0 \\ a & 3 & b \\ c & 1 & d \end{pmatrix}$

3. Determine which of the following are subspaces of $\mathbb{P}_6$.

a. The set, C, of all constant polynomials, $p(t) = c$, $c \in \mathbb{R}$.

b. $T = \{a + bt^2 \mid a, b \in \mathbb{R}\}$

c. The set, V, of polynomials of degree at most 6.

4. Let $A = \begin{pmatrix} 2 & 5 & 4 & 1 & 6 \\ 0 & 5 & 5 & 1 & -1 \\ 0 & 5 & 4 & 1 & -2 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$.

Find bases for and compute the dimension of each of the following spaces:

a. $\mathrm{Null}(A)$

b. $\mathrm{Col}(A)$

c. $\mathrm{Row}(A)$

5. Let A be an $n \times n$ matrix. Write three statements that are equivalent to the statement A is an invertible matrix.

6. Let $\vec{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix}$.

a. Give and example of a vector $\vec{v}_3$ so that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

b. Give and example of a vector $\vec{v}_4$ so that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ is linearly dependent.

7. A system of equations has an augmented matrix of the form:

$$\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & 0 \end{bmatrix}$$

a. Is the system consistent?

b. Say as much as you can about the solution set.

8. Find the matrix for the transformation of $\mathbb{R}^2$ that rotates counterclockwise by $\frac{\pi}{3}$ radians about the origin.

9. Find the 3×3 matrix that produces the following transformation of $\mathbb{R}^2$ using homogeneous coordinates: First reflect across the line $y = x$, then shift up by 2.

10. Which of the following matrices are invertible? Use as few calculations as possible.

a. $\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 & -3 \\ 1 & 1 & 4 \\ 6 & 3 & -9 \end{bmatrix}$

11. Let $B = LU$, where $L = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$, and $U = \begin{bmatrix} 5 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Use this LU factorization of B to solve the matrix equation $B\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$. This problem should be solved without the use of technology.

12. Is the map $\mathbb{R}^2 \to \mathbb{R}^2$ given by $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$ a linear transformation? Explain.

13. Three people sit down to play a game. Each round, the first person give 10% of his money to the second person, the second person gives 20% of his money to the third person, and the third person gives 30% of his money to the first person. Assume they can keep track of fractions of pennies to as many decimal places as needed.

a. Set up the stochastic matrix for this game.

b. If each player starts with the same amount of money, what will the situation be after 10 rounds of the game?

c. Is there a steady state for this situation? If so, find it.

14. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

a. Find the characteristic equation for the A.

b. Use the characteristic equation to find the eigenvalues of A.

15. Let $B = \begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & 4 & 7 \end{bmatrix}$

a. Find the eigenvalues for B.

b. For each eigenvalue λ, find the set of eigenvectors with eigenvalue λ.

16. Let $P = \begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$, and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and let $A = PDP^{-1}$. Compute A^{11}. Please demonstrate how to do this problem without Maple to receive credit.

17. For each of the following, decide if the set is linearly independent, orthogonal both or neither.

a. $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\}$

b. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

c. $\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

18. Let $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 4 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 1 \\ -2 \\ -4 \\ 3 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

a. Describe two different methods for showing that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly independent set. One of your methods should require minimal computation.

b. Let W be the subspace of $\mathbb{R}^4$ spanned by $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Find an orthonormal basis for W.

c. Find the orthogonal projection of $\vec{x}$ onto W.

d. Find the distance from $\vec{x}$ to W.

e. Find the dimension of $W^{\perp}$