Math 203, Exam #3 Review Solutions **Fall 2009**

Your final exam will be cumulative, and will cover §1.1-1.5, 1.7-1.10, 2.1-2.5, 2.7,2.8, 3.1,3.2, 4.1-4.6, 4.9, 5.1-5.3, 6.1-6.4, 7.1. In addition to the problems below, you should look over your homework - **including the true/false questions**, and the review sheets from each of the 3 previous exams. **Be prepared to justify your answers**. Topics we've studied include: Systems of Linear Equations using Matrices, Echelon Form and Reduced Echelon Form, Vector Equations and Matrix Equations, Applications of Linear Algebra Linear Independence, Linear Transformations, Matrix Algebra, the Inverse of a Matrix, Characteristics of Invertible Matrices: Statements that are equivalent to "A is an invertible matrix," Partitioned Matrices LU Factorization, Homogeneous Coordinates, Subspaces of $\mathbb{R}^n$, Determinants, Vector Spaces, Subspaces, Spanning set, Bases, Null Space, Column Space, Row Space, Change of coordinates, Dimension, Rank. Markov Chains, Eigenvectors and Eigenvalues, The Characteristic Equation, Diagonalization, Inner Product, Length, Orthogonal Sets, Orthogonal Projection. Orthonormal sets, The Gram-Schmidt Process, Symmetric Matrices

Here are some problems from §4.9 on. Be sure you know how to do these problems without Maple.

1. A particular strain of flu is going around campus. There are students who have never had the flu, students who have the flu now, and students who are recovered from the flu and now immune. Each week, 5% of the students who never had the flu will get the flu and 20% of the student with flu will recover and become immune.

a. Set up the stochastic matrix for this situation.

$$A = \begin{bmatrix} .95 & 0 & 0 \\ .05 & .8 & 0 \\ 0 & .2 & 1 \end{bmatrix}$$

b. If we start with 80% of the students never having had the flu, and 20% having the flu now, what will the situation be after 10 weeks assuming the trend continues?

Initial state is represented by $\vec{x} = \begin{bmatrix} .8 \\ .2 \\ 0 \end{bmatrix}$

After 10 weeks, the state vector will be $A^{10}\vec{x} = \begin{bmatrix} .4790 \\ .1525 \\ .3685 \end{bmatrix}$

47.90% never had flu, 15.25% have flu and 36.85% immune.

c. Is there a steady state for this situation? If so, find it.

Yes. $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a steady state.

2. Each month, 10% of the employed people lose their job, and half of the unemployed people get a job.

a. Set up the stochastic matrix for this situation.

$B = \begin{bmatrix} .9 & .5 \\ .1 & .5 \end{bmatrix}$

b. Starting with 7% unemployment, what will the situation be in one year, assuming this trend continues?

$\vec{x} = \begin{bmatrix} .93 \\ .07 \end{bmatrix}$, $B^{12}\vec{x} = \begin{bmatrix} 0.8333 \\ 0.1667 \end{bmatrix}$ $\boxed{\text{16.67\% unemployment}}$

c. Is there a steady state for this situation? If so, find it.

Solve: $(B - I)\vec{x} = \vec{0}$ by row reducing $B - I$:

$$B - I = \begin{bmatrix} -.1 & .5 \\ .1 & -.5 \end{bmatrix} \to \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$$

$x_1 = 5x_2$. Then since $x_1 + x_2 = 1$, $\vec{x} = \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix} \approx \begin{bmatrix} 0.8333 \\ 0.1667 \end{bmatrix}$

Note that the system converges to its steady state.

3. Is $\lambda = 10$ an eigenvalue for $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 2 \\ 5 & 3 & 5 \end{bmatrix}$? If so describe the eigenvectors with eigenvalue 10.

$\det(M - 10I) = \det \begin{bmatrix} -9 & 2 & 3 \\ 4 & -5 & 2 \\ 5 & 3 & -5 \end{bmatrix} = 0$, so $\lambda = 10$ is an eigenvalue. To find the corresponding eigenvectors, find the nonzero solutions to $(M - 10I)\vec{x} = \vec{0}$:

$$M - 10I = \begin{bmatrix} -9 & 2 & 3 \\ 4 & -5 & 2 \\ 5 & 3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{19}{37} \\ 0 & 1 & -\frac{30}{37} \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, $x_2 = \frac{30}{37}x_3$, $x_1 = \frac{19}{37}x_3$, $\vec{x} = x_3 \begin{bmatrix} \frac{19}{37} \\ \frac{30}{37} \\ 1 \end{bmatrix} = \frac{x_3}{37} \begin{bmatrix} 19 \\ 30 \\ 37 \end{bmatrix}$. The 10-eigenvectors are the vectors $\left\{ \alpha \begin{bmatrix} 19 \\ 30 \\ 37 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}$

4. Is $\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$ an eigenvector for $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$? If so, find the corresponding eigenvalue.

$\begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 35 \\ 30 \\ 25 \end{bmatrix} = 5 \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix}$. Yes, the eigenvalue is $\lambda = 5$.

5. Find the eigenvalues for $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}$. For each eigenvalue, find an eigenvector.

The eigenvalues are $\lambda = 2, 1, -1$.

$\underline{\lambda = 2}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & -1 & 0 \\ 3 & 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & -1 & 0 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, $x_2 = \frac{3}{5}x_3$, $x_1 = \frac{1}{3}x_2 = \frac{1}{5}x_3$. One example is $\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$. (The 2-eigenspace is generated by this vector).

$\underline{\lambda = 1}$

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 0 & 0 \\ 3 & 4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, $x_2 = \frac{1}{2}x_3$, $x_1 = 0$. One example is $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. (The 1-eigenspace is generated by this vector).

$\underline{\lambda = -1}$

$$\begin{bmatrix} 3 & 0 & 0 \\ 3 & 2 & 0 \\ 3 & 4 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_3 is free, $x_2 = 0$, $x_1 = 0$. One example is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (The 2-eigenspace is generated by this vector).

6. Find the characteristic equation for the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ -3 & 2 & 2 \\ 1 & 5 & 1 \end{bmatrix}$.

$$\det(A - \lambda I) = (2-\lambda)((2-\lambda)(1-\lambda) - 10) - (-3(1-\lambda) - 2) + 3(-15 - (2-\lambda)) = 0$$

$$\boxed{-62 + 2\lambda + 5\lambda^2 - \lambda^3 = 0}$$

7. Find the characteristic equation for the matrix $\begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$, and use it to find the eigenvalues of this matrix.

$(2-\lambda)(4-\lambda) - 6 = 0 \Longrightarrow \lambda^2 - 6\lambda + 2 = 0$. $\boxed{\lambda = 3 \pm \sqrt{7}}$

8. Let $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, and let $A = PDP^{-1}$. Compute A^8.

$$A^8 = (PDP^{-1})^8 = PD^8P^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -509 & 1530 \\ -255 & 766 \end{bmatrix}$$

9. Let $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$.

a. Compute $\vec{v} \cdot \vec{w}$, and $||\vec{v}||$.

$\vec{v}\cdot\vec{w} = -3$, $||\vec{v}|| = \sqrt{14}$.

b. Find a unit vector in the direction of $\vec{v}$.

$\dfrac{\vec{v}}{||\vec{v}||} = \dfrac{1}{\sqrt{14}}\begin{bmatrix}1\\-2\\3\end{bmatrix}$

c. Find the distance between $\vec{v}$ and $\vec{w}$.

$||\vec{v}-\vec{w}|| = \sqrt{30}$

d. Find a nonzero vector that is orthogonal to $\vec{v}$.

One such is $\vec{x} = \begin{bmatrix}2\\1\\0\end{bmatrix}$

10. Decide which of the following sets are orthogonal.

a. $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right\}$ Orthogonal

b. $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ Not.

Orthogonal vectors are linearly independent, so there cannot be a set of 4 vectors in R^3 that are orthogonal.

11. Give an example of an orthonormal basis for $\mathbb{R}^3$, other than the standard basis.

Any three orthogonal vectors in $\mathbb{R}^3$ form an orthogonal basis, so the set $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\end{bmatrix}\right\}$ is an orthogonal basis. To get an orthonormal basis, we have to normalize each of these vectors:

$$\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\-1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{8}}\begin{bmatrix}2\\0\\2\end{bmatrix}\right\}$$

12. Let $\vec{u}_1 = \begin{bmatrix}1\\-4\\-1\\0\end{bmatrix}, \vec{u}_2 = \begin{bmatrix}5\\1\\1\\3\end{bmatrix}, \vec{u}_3 = \begin{bmatrix}0\\1\\-4\\1\end{bmatrix}$.

a. Show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a linearly independent set. There is a way to do this with minimal computation.

This is an orthogonal set of vectors, so the vectors are linearly independent.

b. Let W be the span of $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. Find an orthonormal basis for W.

$\{\frac{1}{\sqrt{18}}\vec{u}_1, \frac{1}{6}\vec{u}_2, \frac{1}{\sqrt{18}}\vec{u}_3\}$

c. Let $\vec{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. Find the orthogonal projection of $\vec{x}$ onto W.

$$\text{proj}_W\vec{x} = \sum_{i=1}^{3} \alpha_i \vec{u}_i, \text{ where } \alpha_i = \frac{\vec{x}\cdot\vec{u}_i}{\vec{u}_i\cdot\vec{u}_i}.\ \alpha_1 = \frac{-4}{18} = -\frac{2}{9}, \alpha_2 = \frac{10}{36} = \frac{5}{18}, \alpha_3 = \frac{-2}{18} = -\frac{1}{9}$$

$$\begin{aligned}\text{proj}_W\vec{x} &= -\frac{2}{9}\vec{u}_1 + \frac{5}{18}\vec{u}_2 - \frac{1}{9}\vec{u}_3 \\ &= \begin{bmatrix} -\frac{2}{9} + \frac{25}{18} \\ \frac{8}{9} + \frac{5}{18} - \frac{1}{9} \\ \frac{2}{9} + \frac{5}{18} + \frac{4}{9} \\ \frac{15}{18} - \frac{1}{9} \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 21 \\ 19 \\ 17 \\ 13 \end{bmatrix}\end{aligned}$$

d. Find the distance from $\vec{x}$ to W.

$$||\vec{x} - \text{proj}_W\vec{x}|| = \sqrt{\left(\frac{3}{18}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{1}{18}\right)^2 + \left(\frac{5}{18}\right)^2} = \boxed{\frac{1}{3}}$$

e. Find the dimension of $W^\perp$

W is in $\mathbb{R}^4$ and W has dimension 3, so $\dim W^\perp = 4 - 3 = \boxed{1}$

f. Find a basis for $W^\perp$.

Since the dimension of $W^\perp$ is 1, any nonzero vector in $W^\perp$ is a basis. In particular,

$$\vec{w} = \vec{x} - \text{proj}_W\vec{x} = \tfrac{1}{18}\begin{bmatrix} -3 \\ -1 \\ 1 \\ 5 \end{bmatrix}$$

13. Use the Gram-Schmidt process to find an orthonormal basis for the span of the vectors:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Start with $\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$. Project $\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ onto $\vec{v_1}$:

$$\hat{y} = \tfrac{5}{7}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \longrightarrow \vec{y} - \hat{y} = \begin{bmatrix} \frac{9}{7} \\ -\frac{3}{7} \\ \frac{5}{7} \\ \frac{2}{7} \end{bmatrix} \longrightarrow \vec{v_2} = \begin{bmatrix} 9 \\ -3 \\ 5 \\ 2 \end{bmatrix}$$

Now project $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ onto $\vec{v_1}$ and $\vec{v_2}$:

$$\hat{x} = \tfrac{1}{7}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + \tfrac{16}{119}\begin{bmatrix} 9 \\ -3 \\ 5 \\ 2 \end{bmatrix} \longrightarrow \vec{x} - \hat{x} = \tfrac{1}{119}\begin{bmatrix} -42 \\ 14 \\ 56 \\ 70 \end{bmatrix} \longrightarrow \vec{v_3} = \begin{bmatrix} -42 \\ 14 \\ 56 \\ 70 \end{bmatrix}$$

Orthogonal basis: $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ -3 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -42 \\ 14 \\ 56 \\ 70 \end{bmatrix}$

Orthonormal basis: $\boxed{\left\{ \frac{1}{7} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{119}} \begin{bmatrix} 9 \\ -3 \\ 5 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{9996}} \begin{bmatrix} -42 \\ 14 \\ 56 \\ 70 \end{bmatrix} \right\}}$

14. Find the QR Factorization of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

In problem 13, we found an orthonormal basis for the column space of this matrix. Use this to get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{9}{\sqrt{119}} & -\frac{42}{\sqrt{9996}} \\ \frac{2}{\sqrt{7}} & -\frac{3}{\sqrt{119}} & \frac{14}{\sqrt{9996}} \\ -\frac{1}{\sqrt{7}} & \frac{5}{\sqrt{119}} & \frac{56}{\sqrt{9996}} \\ \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{119}} & \frac{70}{\sqrt{9996}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{7}} & \frac{9}{\sqrt{119}} & -\frac{3}{\sqrt{51}} \\ \frac{2}{\sqrt{7}} & -\frac{3}{\sqrt{119}} & \frac{1}{\sqrt{51}} \\ -\frac{1}{\sqrt{7}} & \frac{5}{\sqrt{119}} & \frac{4}{\sqrt{51}} \\ \frac{1}{\sqrt{7}} & \frac{2}{\sqrt{119}} & \frac{5}{\sqrt{51}} \end{bmatrix}$$

Solving $A = QR$ for R we get $R = Q^T A$ (since Q is orthonormal).

$$R = \begin{bmatrix} \sqrt{7} & \frac{5}{\sqrt{7}} & \frac{1}{\sqrt{7}} \\ 0 & \frac{\sqrt{119}}{7} & \frac{16}{\sqrt{119}} \\ 0 & 0 & \frac{2\sqrt{3}}{\sqrt{17}} \end{bmatrix}$$

15. Orthogonally diagonalize the matrix $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$.

Find the eigenvectors to get:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix},$$

Find corresponding eigenvectors (of length 1) to get:

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

Then $P^{-1} = P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, and you can check that $PDP^T = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$.

You may get different matrices depending on the order that you write your eigenvalues/eigenvectors.