

1. Determine whether the vector field $\vec{F} = x^2y\vec{i} - xy^2\vec{j}$ is path independent.
 $\text{curl}\vec{F} = -y^2 - x^2 \neq 0$, so \vec{F} is not path independent
2. Let $\vec{F} = 2x^2\vec{i} - 3y\vec{j}$ and let C be the curve parameterized by $\vec{r}(t) = t \cos t \vec{i} + t \sin t \vec{j}$, $0 \leq t \leq 2\pi$. Compute $\int_C \vec{F} \cdot d\vec{r}$. Computing directly:
 $\vec{F}(\vec{r}(t)) = 2(t \cos t)^2 \vec{i} - 3t \sin t \vec{j}$
 $\vec{r}'(t) = (\cos t - t \sin t) \vec{i} + (\sin t + t \cos t) \vec{j}$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F} \cdot \vec{r}'(t) dt = \int_0^{2\pi} 2(t \cos t)^2 (\cos t - t \sin t) + 3t \sin t (\sin t + t \cos t) dt \\ &= \frac{16\pi^3}{3} \end{aligned}$$

3. Let C be the circle of radius 2 centered at $(0,0,0)$, and let $\vec{F} = xy\vec{i} + xy\vec{j}$. Use Green's Theorem to compute $\int_C \vec{F} \cdot d\vec{r}$.

Let S be the disk of radius 2 centered at the origin. Then

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_S \text{curl}\vec{F} dA \\ &= \int_S y - x dA \\ &= 0 \end{aligned}$$

[This integral can be computed in either cartesian or polar coordinates. Or you can observe that $\int_S y dA = 0$ because y is symmetric, and positive for half of the region and negative on the other. $\int_S -x dA = 0$ for the same reason.]

4. Let $\vec{F} = \vec{i} + y\vec{j} + z\vec{k}$. Let S_1 be the surface of the cube $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$. Let S_2 be the cylinder of radius 5 from $z = 0$ to $z = 7$ (not including the top and bottom). Let S_3 be the surface defined by $z = 4x^2 + xy^2$ over the rectangle $-2 \leq x \leq 2$, $0 \leq y \leq 5$.

- a. Compute $\int_{S_1} \vec{F} \cdot d\vec{A}$

This can be done directly or using the Divergence Theorem. Let W be the solid cube determined by $0 \leq x \leq 2$, $0 \leq y \leq 2$, $0 \leq z \leq 2$. Using the Divergence Theorem:

$$\int_{S_1} \vec{F} \cdot d\vec{A} = \int_W \text{div}\vec{F} dV = \int_W 2 dV = 16$$

Alternatively, computing directly:

The flux across the $x = 0$ and the $x = 2$ surfaces cancel each other out, since the \vec{i} component of \vec{F} is constant, and the surface is oriented outward. The flux across the $y = 0$ surface is zero, since the \vec{j} component of \vec{F} is zero on this surface. On the surface $y = 2$, the \vec{j} component of \vec{F} is 2, so the flux across the $y = 2$ surface is 8. Similarly, the flux across the $z = 0$ and $z = 2$ surfaces are 0 and 8, respectively. $\therefore \int_{S_1} \vec{F} \cdot d\vec{A} = 8 + 8 = 16$

- b. Compute $\int_{S_2} \vec{F} \cdot d\vec{A}$

Use the formula:

$$\int_{S_2} \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_a^b \vec{F}(R, \theta, z) \cdot (\cos \theta \vec{i} + \sin \theta \vec{j}) R dz d\theta$$

$\vec{F}(R, \theta, z) = \vec{i} + 5 \sin \theta \vec{j} + z \vec{k}$, so the integrand is $(\cos \theta + 5 \sin^2 \theta)5$

$$\int_{S_2} \vec{F} \cdot d\vec{A} = \int_0^{2\pi} \int_0^7 5 \cos \theta + 25 \sin^2 \theta dz d\theta = 175\pi$$

c. Compute $\int_{S_3} \vec{F} \cdot d\vec{A}$

Use the formula

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x, y, f(x, y)) \cdot (-f_x \vec{i} - f_y \vec{j} + \vec{k}) dx dy$$

$$\vec{F}(x, y, f(x, y)) = \vec{i} + y \vec{j} + (4x^2 + xy^2) \vec{k} \quad \text{and} \quad -f_x \vec{i} - f_y \vec{j} + \vec{k} = -(8x + y^2) \vec{i} - 2xy \vec{j} + \vec{k}$$

so the integrand is $-(8x + y^2) - 2xy^2 + (4x^2 + xy^2) = -8x - y^2 - 2xy^2 + 4x^2 + xy^2$

$$\int_S \vec{F} \cdot d\vec{A} = \int_0^5 \int_{-2}^2 -8x - y^2 - 2xy^2 + 4x^2 + xy^2 dx dy = -60$$

5. $\vec{F} = x^2 z \vec{i} + xy^2 \vec{j} + yz^2 \vec{k}$, and let S be the surface of a sphere of radius 4 centered at $(0,0,0)$. Use the divergence theorem to compute $\int_S \vec{F} \cdot d\vec{A}$

$$\text{div} \vec{F} = 2xz + 2xy + 2yz$$

This can be integrated over the sphere in cartesian coordinates:

$$\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} 2(xz + xy + yz) dz dy dx = 0$$

or using spherical coordinates:

$$\int_0^\pi \int_0^{2\pi} \int_0^4 2(\rho \sin \phi \cos \theta)(\rho \cos \phi) + 2(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta) + 2(\rho \sin \phi \sin \theta)(\rho \cos \phi) d\rho d\theta d\phi$$

$$= 2 \int_0^\pi \int_0^{2\pi} \int_0^4 \rho^2 \sin \phi \cos \phi (\sin \theta + \cos \theta) + \rho^2 \sin^2 \phi \cos \theta \sin \theta d\rho d\theta d\phi = 0$$

6. Let \vec{F} be a vector field, defined at the point (x, y, z) . State the definition of $\text{circ}_{\vec{n}} \vec{F}(x, y, z)$. State the definition of $\text{curl} \vec{F}$.

Let \vec{n} be a vector. Let C be a circle centered at the point (x, y, z) in the plane containing perpendicular to \vec{n} . Then

$$\text{circ}_{\vec{n}} \vec{F}(x, y, z) = \lim_{\text{Area} \rightarrow 0} \frac{\int_C \vec{F} \cdot d\vec{r}}{\text{Area inside } C}$$

$\text{curl} \vec{F}$ is a vector field with the property that at any point (x, y, z) :

- The direction of the vector is the direction \vec{n} that maximizes $\text{circ}_{\vec{n}} \vec{F}(x, y, z)$.
- The magnitude of the vector is this maximum value of $\text{circ}_{\vec{n}} \vec{F}(x, y, z)$.

7. Let $\vec{F} = xe^y\vec{i} + ye^z\vec{j} - ze^x\vec{k}$, and let C be the curve in the plane $z = 2y$ parametrized by $x = \cos t$, $y = \sin t$, $z = 2 \sin t$, $0 \leq t \leq 2\pi$. Use Stokes' Theorem to compute $\int_C \vec{F} \cdot d\vec{r}$.

First, we can describe the surface, S whose boundary is C by:

$$z = 2y, \text{ with } -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}.$$

Use the formula:

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x, y, f(x, y)) \cdot (-f_x\vec{i} - f_y\vec{j} + \vec{k}) dx dy$$

$$\vec{F}(x, y, f(x, y)) = xe^y\vec{i} + ye^{2y}\vec{j} - 2ye^x\vec{k} \quad \text{and} \quad -f_x\vec{i} - f_y\vec{j} + \vec{k} = -2\vec{j} + \vec{k},$$

so the integrand is $-2ye^{2y} + 2ye^x$

$$\int_S \vec{F} \cdot d\vec{A} = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -2ye^{2y} + 2ye^x dy dx \approx -4.3288$$

8. Determine whether the vector field $\vec{F} = x\vec{i} + xy\vec{j} + z\vec{k}$ is the gradient of some function f .
 $\text{curl} \vec{F} = y\vec{k} \neq 0$, so \vec{F} is not the gradient of any function.

9. Determine whether the vector field $\vec{F} = x\vec{i} + 2xz\vec{j} + z\vec{k}$ is the curl of some other vector field.

$\text{div}(\vec{F}) = 1 + 1 \neq 0$, so \vec{F} is not the curl of any vector field.

Extra Problems

1. Let S be the surface of the cone parametrized by $\vec{r}(s, t) = 2s \cos(t)\vec{i} + 2s \sin(t)\vec{j} + (3 - \frac{3s}{2})\vec{k}$, $0 \leq s \leq 2$, $0 \leq t \leq 2\pi$, oriented upward. Let $\vec{F}(x, y) = y\vec{i} + x\vec{j} + x\vec{k}$. Compute $\int_S \vec{F} \cdot d\vec{A}$.

Use the formula:

$$\int_S \vec{F} \cdot d\vec{A} = \int_c^d \int_a^b \vec{F}(\vec{r}(s, t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt$$

$$\vec{F}(\vec{r}(s, t)) = 2s \sin t \vec{i} + 2s \cos t \vec{j} + 2s \cos t \vec{k}$$

$$\begin{aligned} \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) &= \left(2 \cos(t)\vec{i} + 2 \sin(t)\vec{j} + -\frac{3}{2}\vec{k} \right) \times (-2s \sin(t)\vec{i} + 2s \cos(t)\vec{j}) \\ &= 3s \cos t \vec{i} + 3s \sin t \vec{j} + 4s \vec{k} \end{aligned}$$

$$\vec{F}(\vec{r}(s, t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) = 12s^2 \sin t \cos t + 8s^2 \cos t$$

$$\int_0^2 \int_0^{2\pi} 12s^2 \sin t \cos t + 8s^2 \cos t dt ds = \boxed{0}$$

2. Let S be the surface parametrized by $\vec{r}(s, t) = (s + t)\vec{i} + (s - t)\vec{j} + (s^2 - 2t^2)\vec{k}$, $0 \leq s \leq 2$, $0 \leq t \leq 1$, oriented upward. Let $\vec{F}(x, y) = x\vec{i} + y\vec{j} + y\vec{k}$. Compute $\int_S \vec{F} \cdot d\vec{A}$.

Use the formula:

$$\int_S \vec{F} \cdot d\vec{A} = \int_c^d \int_a^b \vec{F}(\vec{r}(s, t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds dt$$

$$\vec{F}(\vec{r}(s, t)) = (s + t)\vec{i} + (s - t)\vec{j} + (s - t)\vec{k}$$

$$\begin{aligned} \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) &= (\vec{i} + \vec{j} + \vec{k}) \times (\vec{i} - \vec{j} - \vec{k}) \\ &= 2\vec{j} - 2\vec{k} \end{aligned}$$

$$\vec{F}(\vec{r}(s, t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) = 2(s - t) - 2(s - t) = 0$$

$$\int \int 0 ds dt = \boxed{0}$$