Math 303, Final **Solutions**
Fall 2005

1. Let $\vec{v} = \vec{\imath} + 2\vec{\jmath} + \vec{k}$, and $\vec{w} = \vec{\imath} - 2\vec{\jmath} + \vec{k}$.

a. Compute $\vec{v} \cdot \vec{w}$.

$1 - 4 + 1 = \boxed{-2}$

b. Compute $\vec{v} \times \vec{w}$.

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \boxed{4\vec{\imath} - 4\vec{k}}$$

c. Find the angle θ between $\vec{v}$ and $\vec{w}$.

Using the fact that $\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}| \cos\theta$:

$$\begin{aligned} -2 &= \sqrt{6}\sqrt{6}\cos\theta \\ &= 6\cos\theta \\ \frac{-2}{6} &= \cos\theta \\ \theta &= \arccos\left(-\frac{1}{3}\right) \approx \boxed{1.910633237 \quad \text{radians}} \end{aligned}$$

2. Let $H(x, y) = x\ln(y)$, and let $\vec{v} = \vec{\imath} + \vec{\jmath}$.

a. Find the Taylor polynomial of degree 2 for H near $(1, 1)$.

$H(1, 1) = 0$
$H_x = \ln y$, $H_x(1, 1) = \ln 1 = 0$
$H_y = \frac{x}{y}$, $H_y(1, 1) = 1$

$\boxed{T = 0 + 0(x - 1) + 1(y - 1) = y - 1}$

b. Compute the directional derivative of $H(x, y)$ in the direction of $\vec{v}$, at the point $(1, 1)$.

$\operatorname{grad} H(1, 1) = H_x(1, 1)\vec{\imath} + H_y(1, 1)\vec{\jmath} = \vec{\jmath}$, $\vec{u} = \frac{1}{\sqrt{2}}(\vec{\imath} + \vec{\jmath})$

$H_{\vec{v}} = \operatorname{grad} H(1, 1) \cdot \vec{u} = \boxed{\dfrac{1}{\sqrt{2}}}$

3. Find the equation for the plane containing the points $(2, 0, 1)$, $(-1, 3, 0)$, and $(4, 4, 0)$.

Let $\vec{v}$ be the vector from $(2, 0, 1)$ to $(-1, 3, 0)$, and let $\vec{w}$ be the vector from $(2, 0, 1)$ to $(4, 4, 0)$:

$$\vec{v} = -3\vec{\imath} + 3\vec{\jmath} - \vec{k}, \qquad \vec{w} = 2\vec{\imath} + 4\vec{\jmath} - \vec{k}$$

Then $\vec{n} = \vec{v} \times \vec{w}$ is normal to this plane.
$\vec{n} = \vec{\imath} - 5\vec{\jmath} - 18\vec{k}$. Then using the point (2,0,1):

$$(x - 2) - 5(y - 0) - 18(z - 1) = 0$$

4. The function $f(x, y)$ is given by

x

		10	12	14
	1	19	20	21
y	2	22	24	26
	3	27	31	35

a. Approximate $f_x(12, 2)$.

Any of the following would be reasonable approximations:
$\frac{26-24}{14-12} = 1$ or $\frac{24-22}{12-10} = 1$ or $\frac{26-22}{14-10} = 1$

b. Approximate $f_y(12, 2)$.

Any of the following would be reasonable approximations:
$\frac{31-24}{3-2} = 7$ or $\frac{24-20}{2-1} = 4$ or $\frac{31-20}{3-1} = \frac{11}{2}$

c. Does $f_{xy}(12, 2)$ appear to be positive, negative or zero? Why?

When $y = 1$, $f_x \approx \frac{1}{2}$. When $y = 2$, $f_x \approx 1$, and when $y = 3$, $f_x \approx 2$, so f_x is increasing as y increases. $\boxed{\therefore f_{xy} \text{ seems to be positive.}}$

d. Find an underestimate for $\displaystyle\int_R f\,dA$, where R is the rectangle $10 \le x \le 14$, $1 \le y \le 3$.

$\Delta x = 2$, $\Delta y = 1$, so the area of each square is 2. For each square, use the smallest function value:
$\sum = 2 \cdot 19 + 2 \cdot 20 + 2 \cdot 22 + 2 \cdot 24 = \boxed{170}$

5. Let $g(x, y) = x^2 y$, $x(u, v) = u^2 v - uv^2$, $y(u, v) = \dfrac{u}{v}$. Use the chain rule to compute the partial derivative g_u. Give your answers in terms of u and v.
$g_u = g_x x_u + g_y y_u = 2xy(2uv - v^2) + x^2(\frac{1}{v}) = 2(u^2 v - uv^2)\frac{u}{v}(2uv - v^2) + (u^2 v - uv^2)^2(\frac{1}{v})$
This simplifies to $u^2 v(u - v)(5u - 3v)$, but that would not have been required.

6. Let $F(x, y) = x^2 y + x - y$. Find all of the critical points. For each critical point decide if it is a local maximum, local minimum, saddle point or none of these.

$F_x = 2xy + 1$ $F_y = x^2 - 1$.
$F_y = 0 \Longrightarrow x = \pm 1$.
For $x = 1$, $2y + 1 = 0$, so $y = -\frac{1}{2}$.
For $x = -1$, $-2y + 1 = 0$, so $y = \frac{1}{2}$.
$\boxed{\text{The critical points are } (1, -\tfrac{1}{2}) \text{ and } (-1, \tfrac{1}{2})}$
$F_{xx} = 2y$, $F_{yy} = 0$, $F_{xy} = 2y$, so $D = -4y^2$. At each critical point $D < 0$, so they are $\boxed{\text{both saddle points.}}$

7. Find the maximum and minimum values of $f(x, y) = 3x^4 + y^4$, subject to the constraint $g(x, y) = x^2 + y^2 \le 4$.

$\text{grad}\, f = 12x^3\vec{\imath} + 4y^3\vec{\jmath}$, so the only critical point is $(0, 0)$.
$\text{grad}\, g = 2x\vec{\imath} + 2y\vec{\jmath}$, so setting $\text{grad}\, f = \lambda\, \text{grad}\, g$:
$12x^3 = 2\lambda x$ and $4y^3 = 2\lambda y$.

Solving for λ, $\lambda = 6x^2 = 2y^2 \Longrightarrow x = \pm\frac{1}{\sqrt{3}}y$. Plug into $x^2 + y^2 = 4$: $4 = \frac{1}{3}y^2 + y^2 = \frac{4}{3}y^2$, so $y = \pm\sqrt{3}$, and $x = \pm 1$. This yields 4 points: $(1, \sqrt{3})$, $(1, -\sqrt{3})$, $(-1, \sqrt{3})$, $(-1, -\sqrt{3})$.

Since we divided by x and by y when we solved for λ, we also have to consider the points where $x = 0$ and where $y = 0$: $(0, \pm 2)$, $(\pm 2, 0)$.

Plug all the points into f:

$f(0,0) = 0 \longleftarrow$ minimum
$f(\pm 1, \pm\sqrt{3}) = 12$
$f(0, \pm 2) = 16$
$f(\pm 2, 0) = 48 \longleftarrow$ maximum

8. Sketch the region of integration for each of the following integrals

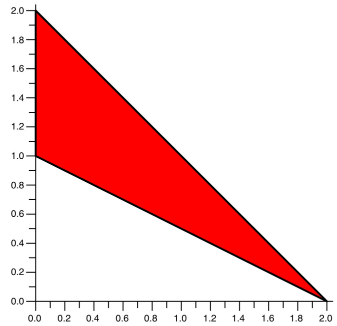
a. $\displaystyle\int_{-1}^{2}\int_{x-2}^{2-x} f(x,y)\,dy\,dx$

b. $\displaystyle\int_{0}^{\pi}\int_{1}^{3} f(r,\theta)\,dr\,d\theta$

c. $\displaystyle\int_{0}^{4}\int_{0}^{2\pi}\int_{0}^{4-z} f(r,\theta,z)\,dr\,d\theta\,dz$

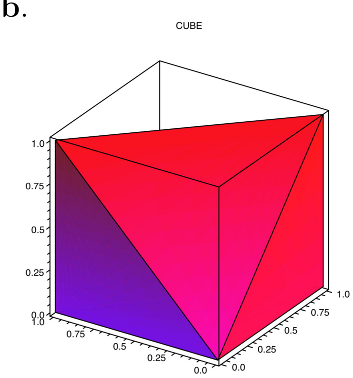
9. For each of the following regions, set up the integral of a function f over the region. State which coordinate system you are using

a.



$$\int_0^2 \int_{1-\frac{1}{2}x}^{2-x} f\, dy\, dx$$

b.



$$\int_0^1 \int_0^{1-x} \int_0^{x+y} f\, dy\, dx$$

10. Let $\vec{F}$ be a three dimensional vector field, f a function of three variables, $\vec{v} = 2\vec{\imath} - 3\vec{\jmath} + 5\vec{k}$ and $a, b,$ and c real numbers. Decide whether each of the following is a number, a point, a function, a vector, or a vector field.

a. curl$\vec{F}$. | vector field |

b. grad $f(a, b, c)$. | vector |

c. $\vec{F} \cdot \vec{v}$ | function |

d. $f_{\vec{v}}$ | $f_{\vec{v}}$ evaluated at a point is a number, $f_{\vec{v}}$ is a function |

e. div$\vec{F}$. | function |

f. div$\vec{F}(a, b, c)$. | number |

11. Is the following statement valid? Explain.

Let $\vec{F}$ be a vector field such that div$\vec{F}(x, y, z) = 0$ *for every point* (x, y, z)*. Then for any surface* S, $\int_S \vec{F} \cdot d\vec{A} = 0$.

12. Let $\vec{F}(x,y) = (x+y)\vec{\imath} + y\vec{\jmath}$

a. Write down the differential equations associated to the vector field.

$\frac{dx}{dt} = x + y,\ \frac{dy}{dt} = y$

b. Determine which of the following parametrized curves could be flow lines for this vector field. For each of the the curves that could be flow lines, show that the curve satisfies the differential equations.

(**i**) $\vec{r}_1(t) = e^{-t}\vec{\imath} + e^t\vec{\jmath}$ Not a flow line.

(**ii**) $\vec{r}_2(t) = te^t\vec{\imath} + e^t\vec{\jmath}$ $\frac{dx}{dt} = \frac{d}{dt}(te^t) = e^t + te^t = x + y\checkmark$ $\frac{dy}{dt} = \frac{d}{dt}(e^t) = e^t = y\checkmark$

(**iii**) $\vec{r}_4(t) = e^t\vec{\imath} + e^{-t}\vec{\jmath}$ Not a flow line.

(**iv**) $\vec{r}_5(t) = 2te^t\vec{\imath} + 2e^t\vec{\jmath}$ $\frac{dx}{dt} = \frac{d}{dt}(2te^t) = 2e^t + 2te^t = x + y\checkmark$ $\frac{dy}{dt} = \frac{d}{dt}(2e^t) = 2e^t = y\checkmark$

13. Let $\vec{F} = 2x\vec{\imath} + 2y\vec{\jmath} + 3xz\vec{k}$.

a. Compute curl$\vec{F}$.

curl$\vec{F} = \boxed{-3z\vec{\jmath}}$.

b. Compute div$\vec{F}$.

$\boxed{\text{div}\vec{F} = 4 + 3x}$

c. Could $\vec{F}$ be the curl of some other vector field? Explain.

No: if $\vec{F} = \text{curl}\vec{G}$, then $\text{div}\vec{F} = \text{div}(\text{curl}\vec{G}) = 0$

d. Compute $\int_C \vec{F}\cdot d\vec{r}$, where C is the circle of radius 2 in the xy-plane, oriented clockwise with respect to the positive z-axis.

Let S be the disk of radius 2 in the xy-plane, then
$\int_C \vec{F}\cdot d\vec{r} = -\int_S \text{curl}\vec{F}\cdot d\vec{A} = \boxed{0}$ (since curl$\vec{F}$ has no $\vec{k}$ component).
Although it isn't important in this case, the minus sign is because the circle is oriented clockwise.

Alternatively, a direct computation using $\vec{r}(t) = -2\cos t\vec{\imath} + 2\sin t\vec{\jmath}$ yields:
$\vec{F}(\vec{r}(t)) = 2(-2\cos t)\vec{\imath} + 2(2\sin t)\vec{\jmath} + 0\vec{k} = -4\cos t\vec{\imath} + 4\sin t\vec{\jmath}$,
$\vec{r}\,'(t) = -2\cos t\vec{\imath} + 2\sin t\vec{\jmath} = 2\sin t\vec{\imath} + 2\cos t\vec{\jmath}$
$\vec{F}(\vec{r}(t))\cdot\vec{r}(t) = 0$, so
$\int_C \vec{F}\cdot d\vec{r} = 0$

14. Let $\vec{F} = y\vec{\imath} - x\vec{k}$. Determine whether each of the following is positive, negative or zero. Explain, or make a sketch to illustrate your answer.

a. $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve $\vec{r}(t) = \cos t\vec{\imath} + \sin t\vec{\jmath}$, $0 \le t \le \pi$.

This curve lies in the xy-plane so the $\vec{k}$ component of $\vec{F}$ can be ignored. The curve lies in the positive y half-plane, and travels in the negative x direction. The vector field points in the positive x direction. $\boxed{\text{negative}}$

b. $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve $\vec{r}(t) = \cos t\vec{\imath} + \sin t\vec{\jmath}$, $0 \le 2\pi$.

Again, we can ignore the $\vec{k}$ component of $\vec{F}$. For the part of the curve in the positive y half-plane the curve travels in the negative x direction, in the negative y half-plane the curve travels in the positive x direction. For the part of the curve in the positive y half-plane the vector field points in the positive x direction. For the part of the curve in the negative y half-plane he vector field points in the negative x direction. $\boxed{\text{negative}}$

Alternatively, use Stoke's Theorem and the fact that curl$\vec{F} = \vec{\jmath} - \vec{k}$ has a negative $\vec{k}$ component.

c. $\int_S \vec{F} \cdot d\vec{A}$, where S is the rectangle $z = 0$, $-1 \le x \le 1$, $0 \le y \le 1$, oriented upward.

Here we need only consider the $\vec{k}$ component of $\vec{F}$. For the negative x half of this rectangle, the vector field points down, for the positive half the vector field points up. By symmetry the integral is $\boxed{0}$.

d. $\int_S \vec{F} \cdot d\vec{A}$, where S is the rectangle $y = 0$, $0 \le z \le 1$, $-1 \le x \le 1$, oriented toward the positive y-axis.

Only consider the $\vec{\jmath}$ component, which is 0, so the integral is $\boxed{0}$.

e. $\int_S \vec{F} \cdot d\vec{A}$, where S is the rectangle $x = 0$, $0 \le z \le 1$, $0 \le y \le 1$, oriented toward the positive x-axis.

Only consider the $\vec{\imath}$ component of the vector field, which is positive. $\boxed{\text{positive}}$.

15. Give a careful statement of the Divergence Theorem.

[The Divergence Theorem] *Let S be a closed surface, oriented outward, and let W be the solid enclosed by S. Let $\vec{F}$ be a differentiable vector field on W and S. Then*

$$\int_S \vec{F} \cdot d\vec{A} = \int_W \text{div}\vec{F}\, dV.$$

16. Let $\vec{F} = 2xz\vec{\imath} - 3x^2y\vec{\jmath} + 2yz^2\vec{k}$. Let S be a cylinder of radius 4, centered on the z-axis with $0 \le z \le 2$. Use the Divergence Theorem to compute $\int_S \vec{F} \cdot d\vec{A}$.

$\text{div}\vec{F} = 2z - 3x^2 + 4yz$

$\text{div}\vec{F}(r,\theta,z) = 2z - 3(r\cos\theta)^2 + 4(r\sin\theta)z = 2z - 3r^2\cos^2\theta + 4rz\sin\theta$

$$\int_S \vec{F}\cdot d\vec{A} = \int_0^2\int_0^4\int_0^{2\pi} 2z - 3r^2\cos^2\theta + 4rz\sin\theta\, d\theta\, dr\, dz = \boxed{32\pi}$$

17. Let $\vec{F} = x\vec{\imath} + z\vec{\jmath}$.

a. Compute $\int_S \vec{F}\cdot d\vec{A}$, where S is the surface of the rectangular box $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 1$, oriented outward.

Use the divergence theorem: $\text{div}\vec{F} = 1$. Integrating over a box with volume 2:

$$\int\int\int 1dV = \boxed{2}$$

b. Compute $\int_S \vec{F}\cdot d\vec{A}$, where S is the part of the surface $f(x,y) = x^2$ over the rectangle $0 \le x \le 3$, $0 \le y \le 2$.

$\vec{F}\cdot(-f_x\vec{\imath} - fy\vec{\jmath} + \vec{k}) = -2x^2$

$$\int\int -2x^2\, dy\, dx = \boxed{-\frac{4}{3}}$$

c. Compute $\int_S \vec{F}\cdot d\vec{A}$, where S is the cylinder of radius 4 centered on the z-axis with $0 \le z \le 3$.

$\vec{F}(4,\theta,z)\cdot(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})4 = (4\cos\theta\vec{\imath} + z\vec{\jmath})\cdot(\cos\theta\vec{\imath} + \sin\theta\vec{\jmath})4 = 16\cos^2\theta + 4z\sin\theta$

$$\int_0^3\int_0^{2\pi} 16\cos^2\theta + 4z\sin\theta\, d\theta\, dz = \boxed{48\pi}$$

18. Let $f(x,y) = 4\sqrt{y^2+4} + x\cos(xy)$. Use the graph of $f(x,y)$ to approximate the maximum and minimum values of $f(x,y)$, subject to the constraint $g(x,y) = x^2 + y^2 \le 4$.

Max: ≈ 12 at $\approx(-\sqrt{2}, \pm\sqrt{2})$
Min: ≈ 6 at $\approx(-2, 0)$

19. Let $f(x,y) = y\sin x\cos y$, and $\vec{v} = \vec{\imath} + \vec{\jmath}$.

a. Use the graph of $f(x,y)$ to locate a point (a,b) such that $f(a,b)$ is defined and $\text{grad}\, f(a,b) = \vec{0}$.

There are many, for example: $\boxed{\text{approximately } (\pm 1.6, \pm .8)}$

b. Use the graph of $f(x,y)$ to determine whether $f_{\vec{v}}(2,2)$ is positive, negative or nearly zero.

Very slightly negative, since the surface slopes down in that direction. I would have accepted 0 though.

c. Approximate $\text{grad}\, f(2,2)$.

A decrease y by .2 corresponds to an increase in z by .4, so $f_y \approx -2$ Also, $\text{grad}f$ points in the direction $\approx (.2\vec{\imath} - j)$, so $\boxed{\text{grad}\, f \approx .4\vec{\imath} - 2\vec{\jmath}}$