Research Statement

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One current strategy in noncommutative ring theory is to associate geometric objects to noncommutative algebras. Algebraists have been very successful analyzing primitive ideals by considering them as geometric objects. For example, the primitive ideals of the universal enveloping algebra of an algebraic solvable Lie algebra, $\mathfrak{g}$, are parametrized by the symplectic leaves in the Poisson manifold $\mathfrak{g}^*$, [D]. A geometric focus was also used in [ATVI] to classify algebras with nice homological properties (similar to polynomial rings) in terms of the geometric structure of a collection of graded indecomposable modules. We refer to this geometric philosophy as noncommutative algebraic geometry.

One theme of my research is utilizing geometry in the study of the twist of a polynomial algebra. Such an algebra is a noncommutative analogue of a homogeneous coordinate ring [ATVI]. It is defined more simply as follows. Given a graded k-algebra $A = \oplus A_d$, and a degree 0 automorphism σ of A, we form the twisted algebra A^σ with multiplication defined on homogeneous elements by $a * b = a \cdot \sigma^r(b)$, where $r = \deg a$, and $\cdot$ denotes usual multiplication in A. Now, let S be the polynomial algebra in n variables over the complex numbers. A graded automorphism σ of S is determined by its restriction to S_1, and scalar multiples of σ give rise to isomorphic twisted algebras, so we can take σ to be an automorphism of $\mathbb{P}^{n-1}$. We are interested in the primitive ideal spectrum of the twist S^σ. We offer the following examples as motivation. Hodges and Levasseur proved that the primitive ideals of the quantum group $\mathcal{O}_q(SL_n)$ are parametrized by the symplectic leaves in SL_n in the Poisson structure induced on $\mathcal{O}(SL_n)$, [H]. Joseph extended these results from SL_n to more arbitrary groups, [J]. Since some of the homomorphic images of quantum groups are twists of commutative algebras, it seems reasonable to ask if a similar result holds for such algebras. In [V] M. Vancliff describes the primitive spectra of S^σ when the automorphism σ is semisimple, and finds an analogous relationship between the primitive ideal spectrum and the symplectic leaves. I have extended these results to more general cases when σ is not necessarily semisimple.

Geometry is introduced by embedding the twisted algebra in a family of flat deformations of the polynomial algebra. The multiplication in S^σ induces a Poisson bracket on S, hence a symplectic structure on $\mathbb{C}^n$. Vancliff proves, in the semisimple case, that the primitive ideals of S^σ are parametrized by the symplectic leaves in $\mathbb{C}^n$. Furthermore, the symplectic leaves are shown to be the orbits in $\mathbb{C}^n$ of a certain algebraic group.

We can illustrate the subtleties of the non-semisimple case with an example. Let $S = k[X, Y, Z]$, and

$$\sigma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Then $S^\sigma \cong B = \mathbb{C} < x, y, z > / < xy - yx - x^2,\ xz - zx - xy + x^2,\ yz - zy - y^2 + xz >$. Each $f \in B$ corresponds to an element $f^0 \in S$, for example, $xy \in B$ corresponds to the polynomial $XY^\sigma = X^2 + XY$. I will say that $f \in B$ is σ-invariant if $(f^0)^\sigma = f^0$. It can be shown that every prime ideal contains an irreducible element that is invariant under σ and that these σ-invariant elements are the scalar multiples of x, and of $f_\alpha = \alpha x^2 + 2xz - y^2$. It follows that the primitive ideals of B are $< x >$; $< x, y, z - \lambda >$, $\lambda \in \mathbb{C}$; and $< f_\alpha >$, $\alpha \in \mathbb{C}$. Now, B can be embedded as a generic member in a family $\{B_q\}$ of twists of S, and the multiplication in B induces a Poisson bracket on S, hence a symplectic structure on $\mathbb{A}^3$. The symplectic leaves associated to the bracket are the points $P_\gamma = \{(0, 0, \gamma)\}$, $\gamma \in \mathbb{C}$, the plane $W = \mathcal{V}(x) \backslash \{P_\gamma\}_{\gamma \in \mathbb{C}}$

$= \{(0, \beta, \gamma) | \beta, \gamma \in \mathbb{C}, \beta \neq 0\}$, and quadratic surfaces $V_\alpha = \mathcal{V}(2xz - y^2 + \alpha x^2)$, $\alpha \in \mathbb{C}$. So, in fact the primitive ideals correspond to symplectic leaves. Now, let G be the algebraic subgroup of $\mathrm{GL}(3, \mathbb{C})$ consisting of matrices of the form

$$M(u,t) = \begin{pmatrix} u & -ut & ut^2/2 \\ 0 & u & -ut \\ 0 & 0 & u \end{pmatrix}.$$

Then G acts by right multiplication on $\mathbb{A}^3$, and the orbits of G are the two dimensional leaves together with $P = \cup P_\gamma$, the union of the zero dimensional leaves.

This example imparts the geometric flavor of my research. In my doctoral dissertation, I showed that if σ has elementary Jordan form, then the primitive ideals in the twisted algebra S^σ are parametrized by the symplectic leaves, and these leaves are orbits of an algebraic group. Last summer I received a research grant from Centenary College to spend several weeks in Eugene, Oregon, working with my former advisor, Brad Shelton. During that time I was able to complete the analysis of the more general case, by combining the result for a Jordan block with the work of Vancliff. I found that with a generic condition on the eigenvalues of σ, the primitive ideals in S^σ are parametrized by the symplectic leaves of the Poisson structure determined by σ. Furthermore, these leaves are realized as orbits of an algebraic group.

I am now investigating other twisted algebras. A twist by an automorphism is an example of the more general notion of a twist by a twisting system. For a graded algebra A, a twisting system for A is a collection of additive group automorphisms $\tau = \{\tau_i\}$ of A so that for $b \in A_j, c \in A_k$, $\tau_i(b)\tau_{i+j}(c) = \tau_i(b\tau_j(c))$, [Z]. This is precisely the requirement on τ so that the operation $*$ on A defined by $a * b = a\tau_i(b)$, $a \in A_i$, $b \in A$ defines a ring multiplication on A. A twisting system τ is said to be trivial if $\tau_i = \sigma^i$ for some automorphism σ of A. It can be shown that the only twisting systems of the polynomial algebra are trivial. In fact, there are few known examples of non-trivial twisting systems of any algebra. I am interested in analyzing, and eventually classifying classes of twisting systems, and the corresponding twisted algebras.

References

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