Math 303, Worksheet #19 **Name: Solutions**
Fall 2008

Yesterday we saw the formula for computing the flux through a surface defined by $z = f(x,y)$ with x, y in a region R:

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(x,y,f(x,y)) \cdot (-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k})\,dx\,dy.$$

Then we saw the formula for computing the flux through a surface defined by a parametrization $\vec{r}(s,t)$:

$$\int_S \vec{F} \cdot d\vec{A} = \int_R \vec{F}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) ds\,dt.$$

1. Let S_1 be the surface defined by $z = x^2 - y^2$ with $-2 \le x \le 2$ and $-2 \le y \le 2$, and let $\vec{F} = xy\vec{\imath} - yz\vec{\jmath} + zx\vec{k}$.

a. Compute $\int_S \vec{F} \cdot d\vec{A}$.

$\vec{F}(x,y,f(x,y)) = xy\vec{\imath} - y(x^2-y^2)\vec{\jmath} + x(x^2-y^2)\vec{k}$
$-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k} = -2x\vec{\imath} + 2y\vec{\jmath} + \vec{k}$
$\vec{F}(x,y,f(x,y)) \cdot (-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}) = -2x^2y - 2y^2(x^2-y^2) + x(x^2-y^2)$

$$\int_S \vec{F} \cdot d\vec{A} = \int_{-2}^{2}\int_{-2}^{2} -2x^2y - 2y^2(x^2-y^2) + x(x^2-y^2)\,dx\,dy = \boxed{\frac{2048}{45} \approx 45.511}$$

b. Repeat part **a** using the surface S_2, defined by the same function over the disk with radius 1 centered at the origin.

The only things that change are the limits of integration:

$$\int_S \vec{F} \cdot d\vec{A} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} -2x^2y - 2y^2(x^2-y^2) + x(x^2-y^2)\,dx\,dy = \boxed{\frac{\pi}{6}}$$

2. Let S_3 be the surface parametrized by $\vec{r}(s,t) = s^3t\vec{\imath} + s^2t^2\vec{\jmath} + st^3\vec{k}$

a. Compute $\int_{S_3} \vec{G} \cdot d\vec{A}$ where $\vec{G} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

$\vec{G}(\vec{r}(s,t)) = s^3t\vec{\imath} + s^2t^2\vec{\jmath} + st^3\vec{k}$
$\frac{\partial \vec{r}}{\partial s} = 3s^2t\vec{\imath} + 2st^2\vec{\jmath} + t^3\vec{k}$
$\frac{\partial \vec{r}}{\partial t} = s^3\vec{\imath} + 2s^2t\vec{\jmath} + 3st^2\vec{k}$
$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = 4s^2t^4\vec{\imath} - 8s^3t^3\vec{\jmath} + 4s^4t^2\vec{k}$
$\vec{G}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) = 0$ $\quad \int_R 0\,ds\,dt = \boxed{0}$

b. Repeat part **a** with the vector field $\vec{H} = x\vec{\imath} - y\vec{\jmath} + z\vec{k}$

The only difference is that now $\vec{G}(\vec{r}(s,t)) = s^3t\vec{\imath} - s^2t^2\vec{\jmath} + st^3\vec{k}$, so
$\vec{G}(\vec{r}(s,t)) \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) = 16s^5t^5$.

$$\int_0^5\int_0^5 16s^5t^5\,ds\,dt = \boxed{\frac{976562500}{9} \approx 1.085 \times 10^8}$$

3. On the last worksheet we constructed the formula for flux through a sphere of radius R:

$$\int_S \vec{F} \cdot d\vec{A} = \int_T \vec{F}(R, \phi, \theta) \cdot (\sin\phi\cos\theta\vec{\imath} + \sin\phi\sin\theta\vec{\jmath} + \cos\phi\vec{k})R^2 \sin\phi\, d\phi\, d\theta$$

a. Compute the flux across a sphere S_R of radius R for the vector field $\vec{G} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$. This should simplify nicely.

$\vec{G}(R, \phi, \theta) = R\sin\phi\cos\theta\vec{\imath} + R\sin\phi\sin\theta\vec{\jmath} + R\cos\phi\vec{k} = R(\sin\phi\cos\theta\vec{\imath} + \sin\phi\sin\theta\vec{\jmath} + \cos\phi\vec{k})$

$$\begin{aligned}
&\vec{G}(R, \phi, \theta) \cdot (\sin\phi\cos\theta\vec{\imath} + \sin\phi\sin\theta\vec{\jmath} + \cos\phi) \\
&\quad = R(\sin\phi\cos\theta\vec{\imath} + \sin\phi\sin\theta\vec{\jmath} + \cos\phi\vec{k}) \cdot (\sin\phi\cos\theta\vec{\imath} + \sin\phi\sin\theta\vec{\jmath} + \cos\phi\vec{k}) \\
&\quad = R(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi) \\
&\quad = R[\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi] \\
&\quad = R[\sin^2\phi + \cos^2\phi] \\
&\quad = R
\end{aligned}$$

$$\begin{aligned}
\int_S \vec{G} \cdot d\vec{A} &= \int_T \vec{G}(R, \phi, \theta) \cdot (\sin\phi\cos\theta\vec{\imath} + \sin\phi\sin\theta\vec{\jmath} + \cos\phi\vec{k})R^2 \sin\phi\, d\phi\, d\theta \\
&= \int_T R \cdot R^2 \sin\phi\, d\phi\, d\theta \\
&= R^3 \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\phi\, d\theta \\
&= \boxed{4R^3\pi}
\end{aligned}$$

b. Let V be the volume of S_R. Compute $\dfrac{\int_{S_R} \vec{G} \cdot d\vec{A}}{V}$. There should be a formula for the volume of a sphere in the cover of your text.

$V = \frac{4}{3}\pi R^3$

$$\frac{\int_{S_R} \vec{G} \cdot d\vec{A}}{V} = \frac{4R^3\pi}{\frac{4}{3}\pi R^3} = \boxed{3}$$

4. For a vector field $\vec{F} = F_1\vec{\imath} + F_2\vec{\jmath} + F_3\vec{k}$, the *divergence* of $\vec{F}$ is defined by

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Compute the divergence of each of the following vector fields.

a. $\vec{G} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$1 + 1 + 1 = \boxed{3}$

b. $\vec{F} = xy\vec{\imath} - yz\vec{\jmath} + zx\vec{k}$

$\boxed{y - z + x}$