

# A MANY PERMUTATION GROUP RESULT FOR UNSTABLE THEORIES

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ABSTRACT. We extend Shelah's first many model result to show that an unstable theory has  $2^\kappa$  many non-permutation group isomorphic models of size  $\kappa$ , where  $\kappa$  is an uncountable regular cardinal.

## 1. INTRODUCTION

In [S2], Shelah proved

**Theorem 1.1.** *An unstable countable first order theory has  $2^\kappa$  nonisomorphic models of size  $\kappa$ , where  $\kappa$  is uncountable.*

In this paper, we will consider a strengthening of that result for regular uncountable cardinals. Specifically, we will consider the concept of permutation group isomorphism, where we define two models  $M$  and  $N$  to be permutation group isomorphic if the two associated permutation groups  $\langle |M|, \text{Aut}(M) \rangle$  and  $\langle |N|, \text{Aut}(N) \rangle$  are isomorphic as permutation groups. (Roughly speaking,  $M$  and  $N$  are permutation group isomorphic if the group action on  $M$  by its group of automorphisms is the same as the group action on  $N$  by its group of automorphisms.)

Our goal is to prove the following generalization of Shelah's result:

**Theorem 1.2.** *Given  $T$ , a countable first-order theory, and  $\kappa$ , a regular uncountable cardinal, if  $T$  is unstable, then there exists  $2^\kappa$  many models of  $T$  of size  $\kappa$  which are pairwise non-permutation group isomorphic.*

The strategy we use will be much the same as in the proof of Theorem 1.1: using the instability of the theory  $T$ , we will create  $2^\kappa$  many models from sufficiently different orderings so that the models themselves are not isomorphic. The difficulty in proving the above theorem is to find orderings which are different enough to ensure the corresponding models are not permutation group isomorphic.

## 2. THE BASIC DEFINITIONS

We first start by defining a permutation group:

**Definition 2.1.** A permutation group is a pair  $\langle X, G \rangle$  (where  $X$  is a set and  $G$  is a group) together with an action of  $G$  on  $X$  such that if for all  $x \in X$ ,  $gx = x$ , then  $g = 1$ .

**Definition 2.2.** Given two permutation groups  $\langle X_1, G_1 \rangle$  and  $\langle X_2, G_2 \rangle$ , we say that the two permutation groups are isomorphic if there exists a bijection  $f : X_1 \rightarrow X_2$  such that the map  $g \mapsto fgf^{-1}$  is an isomorphism of the groups  $G_1$  and  $G_2$ .

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(Some authors use the terminology that  $\langle X_1, G_1 \rangle$  and  $\langle X_2, G_2 \rangle$  are similar, not isomorphic. See the opening discussion of [K-M] for details.)

**Proposition 2.3.** *Suppose  $\langle X_1, G_1 \rangle$  and  $\langle X_2, G_2 \rangle$  are isomorphic via a map  $f : X_1 \rightarrow X_2$ . Given  $A \subset X_1, \bar{a}, \bar{b} \in X_1$ , if there exists  $g \in G_1$  such that  $g$  fixes  $A$  pointwise and  $g(\bar{a}) = \bar{b}$ , then there exists  $g' \in G_2$  such that  $g'$  fixes  $f(A)$  pointwise and  $g'(f(\bar{a})) = f(\bar{b})$ .*

*Proof.* Let  $g' = f g f^{-1}$ . □

Put another way, if  $\bar{a}$  and  $\bar{b}$  are in the same orbit over  $A$  in  $X_1$ , then  $f(\bar{a})$  and  $f(\bar{b})$  are in the same orbit over  $f(A)$  in  $X_2$ . We now move to looking at the permutation groups associated with models:

**Definition 2.4.** Given a model  $M$  of a theory  $T$ , the permutation group associated with the model  $M$  is the pair  $\langle |M|, \text{Aut}(M) \rangle$  (where  $|M|$  stands for the universe of the model, and  $\text{Aut}(M)$  stands for the group of automorphisms of  $M$ ) with the group action  $gx$  defined to be  $g(x)$ .

Cherlin and Lachlan have used the term permutation structure for the above concept in [C-L]. On the other hand, Hodges in [H] simply uses the term automorphism group. We have decided to follow the recently published work [K-M] in our choice of nomenclature.

We will refer to elements of  $\text{Aut}(M)$  as  $\sigma$  and  $\tau$  rather than  $g_1$  and  $g_2$ , and will write the group action as  $\sigma(x)$  rather than  $\sigma x$ .

**Definition 2.5.** Given two models  $M$  and  $N$ , we will say that  $M$  is permutation group isomorphic to  $N$  if the permutation group  $\langle |M|, \text{Aut}(M) \rangle$  is isomorphic to  $\langle |N|, \text{Aut}(N) \rangle$ .

The feature of permutation group isomorphisms we will use the most is the following, which is simply Proposition 2.3 restated for models.

**Proposition 2.6.** *Given  $M$  permutation group isomorphic to  $M'$  via a map  $f$ , if  $\bar{a}$  and  $\bar{b}$  are in the same orbit over  $A$  in  $M$ , then  $f(\bar{a})$  and  $f(\bar{b})$  are in the same orbit over  $f(A)$  in  $M'$ .*

### 3. PRELIMINARY FACTS

We state the version of the definition of stability we will need (as found in [S1].)

**Definition 3.1.** A theory  $T$  is unstable if there exists a model  $M$  of  $T$ , a natural number  $n \in \omega$ , an infinite sequence of  $n$ -tuples  $\bar{a}_0, \dots, \bar{a}_m, \dots \in M^n$ , and a formula  $\varphi(\bar{x}, \bar{y}) \in L$  such that for every  $p, q \in \omega$

$$M \models \varphi(\bar{a}_p, \bar{a}_q) \Leftrightarrow p < q$$

We will say that  $\varphi(\bar{x}, \bar{y})$  witnesses the instability of  $T$  if there exists a model  $M$  of  $T$  and an infinite sequence of  $n$ -tuples  $\bar{a}_0, \dots, \bar{a}_m, \dots \in M^n$  such that the conclusion of the above definition is true.

All the models we will be considering will be Ehrenfeucht-Mostowski models as given below:

**Theorem 3.2.** *If  $T$  is an unstable countable Skolem theory in a language  $L$  where  $\varphi(\bar{x}, \bar{y})$  witnesses the instability of  $T$ , then given any linear ordering  $\Delta$  ordered by  $<$ , there exists a model  $M(\Delta)$  containing an ordering of  $n$ -tuples  $A = \{\bar{a}_l\}_{l \in \Delta}$  such that*

1. every element of  $M(\Delta)$  is of the form  $t(\bar{a}_{l_1}, \dots, \bar{a}_{l_m})$  for some  $m \in \omega$ , some term  $t \in L$ , and  $l_1 < \dots < l_m \in \Delta$ ,
2.  $A$  is indiscernible in  $M(\Delta)$ ,
3. for all  $l_1, l_2 \in \Delta$ ,  $M(\Delta) \models \phi(\bar{a}_{l_1}, \bar{a}_{l_2}) \Leftrightarrow l_1 < l_2$ , and
4. every 1-1, onto, order-preserving map  $f : \Delta \rightarrow \Delta$  induces an automorphism of  $M(\Delta)$  by mapping all elements of the form  $t(\bar{a}_{l_1}, \dots, \bar{a}_{l_m})$  (where  $t$  is a term in  $L$ ) to  $t(\bar{a}_{f(l_1)}, \dots, \bar{a}_{f(l_m)})$ .

(Proofs of the above theorem are found throughout the literature; a good exposition can be found in Chapter 11 of [H].)

#### 4. THE $\kappa_e^\omega(X)$ ORDERING

In this section, we will construct a specific ordering that fulfills the following goal:

There exists a dense set of elements in the ordering such that if  $a, b$  are in this set, then given any two increasing tuples in the interval  $(a, b)$  having the same length, there exists an automorphism of the ordering moving one tuple to the other and fixing all elements not in the interval  $(a, b)$ .

We will use the variables  $a, b, c, d, l$  to refer to sequences (i.e., elements of  $\beta^\omega$  for some ordinal  $\beta$ ) and use the notation  $a(i)$  to refer to the  $i$ th entry of  $a$ . Given a finite tuple  $\bar{b} \in \beta^{<\omega}$  and  $a \in \beta^\omega$ , we will let  $\bar{b} \frown a$  be the sequence created by concatenating  $\bar{b}$  and  $a$ .

Throughout this section,  $\kappa$  will be an uncountable (and possibly regular) cardinal. We define our ordering  $\kappa_e^\omega$  to be the subset of  $(\kappa + 1)^\omega$  consisting of all sequences which eventually become constant with value  $\kappa$ . (For the sake of clarity, we will write  $\kappa$  as **e** to indicate that it is the last (or **end**) element.) We give  $\kappa_e^\omega$  the usual lexicographical ordering. To refer to  $n$ -tuples from  $\kappa_e^\omega(X)$ , we will use the notation  $\langle a_1, \dots, a_n \rangle$ . We will often write an element  $a \in \kappa_e^\omega$  in the form  $a = a(0)a(1)a(2)\dots$ .

Often, we will be interested in the case when  $\kappa$  is regular. In such a case, we will want to consider subsets of  $\kappa$  consisting of cofinality  $\omega$  cardinals. Fixing such a set  $X$ , for  $\alpha \in X$ , let  $g_\alpha : \omega \rightarrow \alpha$  be a strictly increasing function such that  $\sup_{n \in \omega} g_\alpha(n) = \alpha$ . (The specific choice of the  $g_\alpha$ 's is arbitrary, but important. We will be using the  $g_\alpha$ 's extensively in the remainder of the paper.) In Shelah's original construction in the many-model proof, it would have been sufficient simply to add  $g_\alpha$  to the ordering. However, to ensure the property mentioned at the beginning of this section, we need to add to our ordering not only  $g_\alpha$ , but also the elements that will be images of  $g_\alpha$  under automorphisms of the ordering. In other words, we need to *propagate*  $g_\alpha$  throughout the ordering.

Let  $prop(X)$  be the set

$$\{g \in (\kappa + 1)^\omega : \exists n_0, n_1 \in \omega \exists \alpha \in X \forall i \in \omega g(n_0 + i) = g_\alpha(n_1 + i)\}.$$

We define  $\kappa_e^\omega(X) = \kappa_e^\omega \cup prop(X)$  ordered lexicographically. When needed,  $\alpha_e^\omega(X)$  will be the subset of  $\kappa_e^\omega(X)$  consisting of sequences each of whose entries equals **e** or is less than  $\alpha$ . We will also be referring to the set  $Aut(\kappa_e^\omega(X))$  of automorphisms of  $\kappa_e^\omega(X)$ . Note that  $\sigma \in Aut(\kappa_e^\omega(X))$  if and only if  $\sigma$  is an 1-1, onto, order-preserving map from  $\kappa_e^\omega(X)$  to  $\kappa_e^\omega(X)$ .

We need the following fact about the size of  $\alpha_e^\omega(X)$ :

**Proposition 4.1.** *Given  $\kappa$ , a regular uncountable cardinal, given a set  $X$  of ordinals all less than  $\kappa$  with cofinality  $\omega$ , and given an ordinal  $\alpha < \kappa$ ,  $\|\alpha_e^\omega(X)\| \leq \|\alpha\| + \aleph_0 < \kappa$ .*

*Proof.* If  $g \in \alpha_e^\omega(X) \cap \kappa_e^\omega$ , then  $g$  consists of an finite initial segment of entries from  $\alpha \cup \{\mathbf{e}\}$  followed by  $\mathbf{eee}\dots$ . It is easy to check that the set of all such  $g$  has cardinality less than or equal to  $\|\alpha\| + \aleph_0$ .

So suppose  $g \in \alpha_e^\omega(X) \cap \text{prop}(X)$ . Then there exists  $n_0, n_1 \in \omega$  and  $\beta \in X$  such that for all  $i \in \omega$ ,  $g(n_0 + i) = g_\beta(n_1 + i)$ . If  $\beta > \alpha$ , then for some  $m$ ,  $g_\beta(m) > \alpha$  (since  $\sup_{n \in \omega} g_\beta(n) = \beta$ .) Since  $g_\beta$  is a strictly increasing function, for all  $m' \geq m$ ,  $g_\beta(m') > \alpha$ . But this implies that for sufficiently large  $n$ ,  $\alpha < g(n) < \mathbf{e}$ . This is impossible since  $g \in \alpha_e^\omega(X)$ . Therefore  $\beta \leq \alpha$ .

In other words, if  $g \in \alpha_e^\omega(X) \cap \text{prop}(X)$ , then  $g \in \text{prop}(X \cap \{\beta : \beta \leq \alpha\})$ . Given this,  $g$  must consist of an initial segment of entries all of whom are either less than  $\alpha$  or equal to  $\mathbf{e}$ , followed by a final segment which is a final segment of some  $g_\beta$  for some  $\beta \in X \cap \{\beta : \beta \leq \alpha\}$ . The total number of such initial segments is  $\|(\alpha \cup \{\mathbf{e}\})^{<\omega}\| = \|\alpha\| + \aleph_0$ . The total number of such final segments is  $\|X \cap \{\beta : \beta \leq \alpha\}\| + \aleph_0 = \|\alpha\| + \aleph_0$  (we must add the  $\aleph_0$  because every  $g_\beta$  for  $\beta \in X \cap \{\beta : \beta \leq \alpha\}$  has at most  $\aleph_0$  many final segments.) Thus  $\|\text{prop}(X \cap \{\beta : \beta \leq \alpha\})\| \leq \|\alpha\| + \aleph_0 < \kappa$  (since  $\kappa$  is uncountable.) This fact along with the fact above that  $\|\alpha_e^\omega(X) \cap \kappa_e^\omega\| \leq \|\alpha\| + \aleph_0$  gives us the desired result.  $\square$

We make the following definitions:

**Definition 4.2.** For  $l \in \kappa_e^\omega(X)$ , let  $ht(l)$  (the height of  $l$ ) be the least value of  $n$  such that for all  $m \geq n$ ,  $l(n) = \mathbf{e}$ , setting  $ht(l) = \omega$  if no such  $n$  exists.

For  $l \in \kappa_e^\omega(X)$ , let  $lh(l)$  equal the number of entries of  $l$  not equal to  $\mathbf{e}$ .

For  $\bar{b} \in (\kappa + 1)^{<\omega}$ ,  $\bar{b}$  being an  $n$ -tuple, let  $\text{slice}(\bar{b}) = \{l \in \kappa_e^\omega(X) \mid l \upharpoonright n = \bar{b}\}$

**Proposition 4.3.** *Given  $\bar{a}, \bar{b} \in (\kappa + 1)^{<\omega}$ , the map  $\tau : \text{slice}(\bar{a}) \rightarrow \text{slice}(\bar{b})$  defined by mapping  $\bar{a} \frown c \in \text{slice}(\bar{a})$  to  $\bar{b} \frown c \in \text{slice}(\bar{b})$  (where  $c$  is a sequence from  $\kappa_e^\omega(X)$ ) is a 1-1, onto, order-preserving map.*

*Proof.* Clearly, the map  $\tau$  is 1-1 and order-preserving. To show that the map is onto, given  $l \in \text{slice}(\bar{b})$ , write  $l$  as  $\bar{b} \frown c$ , where  $c$  is a sequence in  $\kappa_e^\omega(X)$ . If  $\bar{b} \frown c \in \kappa_e^\omega$ , then  $c \in \kappa_e^\omega$ , and thus  $\bar{a} \frown c$  is the desired member of  $\text{slice}(\bar{a})$ .

Suppose  $\bar{b} \frown c \in \text{prop}(X)$ . Note that  $\bar{b} \frown c \in \text{prop}(X)$  if and only if some final segment of  $\bar{b} \frown c$  equals some final segment of  $g_\alpha$  for some  $\alpha \in X$ . Since the same is true of  $\bar{a} \frown c$ , we have that  $\bar{a} \frown c \in \text{prop}(X)$  and our map is onto.  $\square$

Throughout this section, we will be defining automorphisms of  $\kappa_e^\omega(X)$  by listing the action of the automorphisms on slices of  $\kappa_e^\omega(X)$ . In particular, we will use the notation  $\tau(\text{slice}(\bar{a})) = \text{slice}(\bar{b})$  to refer to the above map between  $\text{slice}(\bar{a})$  and  $\text{slice}(\bar{b})$ .

It is helpful to prove the following fact:

**Proposition 4.4.** *Given  $a, b \in \kappa_e^\omega(X)$  such that  $a < b$ , there exists  $c \in \kappa_e^\omega$  such that  $a < c < b$  (i.e.,  $\kappa_e^\omega$  is dense in  $\kappa_e^\omega(X)$ .)*

*Proof.* Since  $a < b$ , there exists  $j$  such that  $a \upharpoonright j = b \upharpoonright j$  and  $a(j) < b(j)$ . If there exists  $\alpha < \kappa$  such that  $a(j) < \alpha < b(j)$ , let  $c = (a \upharpoonright j) \frown \alpha \frown \mathbf{eee}\dots$ . If no such  $\alpha$  exists, find the least  $k > j$  such that  $b(k) \neq 0$ . Then let  $c = (b \upharpoonright k) \frown 0 \frown \mathbf{eee}\dots$ . It is easy to check that in both cases,  $a < c < b$ .  $\square$

We will specifically be interested in maps having the following property:

**Definition 4.5.** Given  $\tau : \kappa_e^\omega(X) \rightarrow \kappa_e^\omega(X)$ , we say  $\tau$  is relatively height preserving if  $\tau$  sends elements of finite height to elements of finite height and elements of infinite height to elements of infinite height. (In other words,  $\tau$  fixes setwise the sets  $\kappa_e^\omega$  and  $\text{prop}(X)$ .)

**Proposition 4.6.** Given  $a \in \kappa_e^\omega - \{\mathbf{eee}\dots\}$ , the interval  $\{l \in \kappa_e^\omega(X) \mid l > a\}$  is isomorphic to  $\kappa_e^\omega(X)$  via an isomorphism  $\tau$  which is relatively height preserving.

*Proof.* We will induct on  $ht(a)$ . Suppose  $ht(a) = 1$ . So  $a$  is of the form  $a(0) \frown \mathbf{eee}\dots$ . We know that the order type of  $\{\alpha \in \text{Ord} \mid a(0) < \alpha < \mathbf{e}\}$  has order type  $\kappa$  via an isomorphism  $f$ . Define  $\tau$  by

$$\tau(\text{slice}(\alpha)) = \begin{cases} \text{slice}(f(\alpha)) & \text{if } a(0) < \alpha < \mathbf{e} \\ \text{slice}(\mathbf{e}) & \text{if } \alpha = \mathbf{e} \end{cases}$$

Suppose  $ht(a) = n$ , and we have proven the case for all smaller values of  $ht(a)$ . So  $a$  is of the form  $a(0)a(1)a(2)\dots a(n-1)\mathbf{eee}\dots$ . By our definition of  $ht(a)$ , we must have that  $a(n-1) \neq \mathbf{e}$ . Again, the order type of  $\{\alpha \in \text{Ord} \mid a(n-1) < \alpha < \mathbf{e}\}$  has order type  $\kappa$  via an isomorphism  $f$ . By our induction hypothesis, the interval  $A = \{l \in \kappa_e^\omega(X) \mid l > a(0)a(1)\dots a(n-2)\mathbf{eee}\dots\}$  is isomorphic to  $\kappa_e^\omega(X)$  via a relatively height preserving map  $\tau'$ . Let  $\sigma : \kappa_e^\omega(X) \rightarrow \kappa_e^\omega(X)$  be defined by  $\sigma(\text{slice}(\alpha)) = \text{slice}(1 + \alpha)$  for all  $\alpha < \mathbf{e}$ . Define  $\tau$  by

$$\begin{aligned} \tau(\text{slice}(a(0)a(1)\dots a(n-2) \frown \alpha)) &= \text{slice}(0 \frown f(\alpha)) \text{ if } a(n-1) < \alpha < \mathbf{e} \\ \tau(\text{slice}(a(0)a(1)\dots a(n-2) \frown \mathbf{e})) &= \text{slice}(0 \frown \mathbf{e}) \\ \tau(c) &= \sigma\tau'(c) \text{ for } c \in A. \end{aligned}$$

Here we are using the 0's and the  $1 + \alpha$  to ensure that we have a bijection.

Note that  $\tau$  is clearly 1-1. To see that  $\tau$  is onto, suppose that  $l \in \kappa_e^\omega(X)$  is a member of  $\text{slice}(\beta)$  for some ordinal  $\beta$ . If  $\beta \neq 0$ , we can write  $l = l(0)l(1)\dots$  and then  $l' = (l(0) - 1) \frown l(1)l(2)\dots$  is our desired preimage. If  $\beta = 0$ , we can write  $l = 0 \frown l(1)l(2)\dots$ , and then  $l' = a(0)\dots a(n-2) \frown f^{-1}(l(1)) \frown l(2)\dots$  is our desired preimage.

Also note that since we are building  $\tau$  from the maps defined in Proposition 4.3, and those maps are order-preserving and relatively height preserving, it is easy to check that  $\tau$  is both order-preserving (and thus a member of  $\text{Aut}(\kappa_e^\omega(X))$ ) and relatively height preserving.  $\square$

**Proposition 4.7.** Given  $b \in \kappa_e^\omega - \{\mathbf{eee}\dots\}$ , the set  $\{l \in \kappa_e^\omega(X) \mid l \leq b\}$  is isomorphic to  $\kappa_e^\omega(X)$  via a relatively height preserving isomorphism  $\tau : \{l \in \kappa_e^\omega(X) \mid l \leq b\} \rightarrow \kappa_e^\omega(X)$ .

*Proof.* We will actually construct an isomorphism  $\tau$  from  $\{l \in \kappa_e^\omega(X) \mid l \leq b\}$  to  $\kappa_e^\omega(X)$  satisfying the following three properties:

1.  $\tau$  is relatively height preserving.
2. For  $b' \in \kappa_e^\omega$ , if  $b' \in \text{slice}(b \upharpoonright ht(b))$  and  $lh(b')$  is finite, then  $lh(b') > lh(\tau(b'))$ . (Note that since  $b' \in \text{slice}(b \upharpoonright ht(b))$ ,  $b'$  is in the domain of  $\tau$ .)
3. Given  $b', b'' \in \text{slice}(b \upharpoonright ht(b))$  with  $ht(b'), ht(b'') < \omega$  and  $b, b', b''$  all different, if  $b'' \in \text{slice}(b' \upharpoonright ht(b'))$ , then  $\tau(b'') \in \text{slice}(\tau(b') \upharpoonright ht(\tau(b')))$ .

We will induct on  $lh(b)$ . First assume that  $lh(b) = 1$ . Suppose  $ht(b) = 1$  so we have  $b = b(0) \frown \mathbf{eee} \dots$ . Our desired  $\tau$  will be

$$\begin{aligned} \tau(\text{slice}(\alpha)) &= \text{slice}(\alpha) \text{ if } \alpha < b(0), \text{ and} \\ \tau(\text{slice}(b(0) \frown \alpha)) &= \text{slice}(b(0) + \alpha) \text{ if } \alpha \leq \mathbf{e}. \end{aligned}$$

It is easy to check that our desired conditions hold for  $\tau$ .

Suppose  $ht(b) = n$  (where  $n > 1$ ) so we have  $b = \mathbf{e} \dots \mathbf{e}b(n-1)\mathbf{eee} \dots$  and  $b(n-1) \neq \mathbf{e}$ . Define  $\tau$  by

$$\begin{aligned} \tau(\text{slice}(\alpha)) &= \text{slice}(0 \frown \alpha) \text{ for all } \alpha < \mathbf{e}, \\ \tau(\text{slice}(\mathbf{e} \frown \alpha)) &= \text{slice}(0 \frown \mathbf{e} \frown \alpha) \text{ for all } \alpha < \mathbf{e}, \end{aligned}$$

$\vdots$

$$\tau(\text{slice}(\overbrace{\mathbf{e} \dots \mathbf{e}}^{n-2} \frown \alpha)) = \text{slice}(0 \frown \overbrace{\mathbf{e} \dots \mathbf{e}}^{n-2} \frown \alpha) \text{ for all } \alpha < \mathbf{e}.$$

If we know  $b(n-1) \neq 0$ , then we complete the definition of  $\tau$  by setting

$$\tau(\text{slice}(\overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1} \frown 0)) = \text{slice}(0 \frown \overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1}),$$

$$\tau(\text{slice}(\overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1} \frown \alpha)) = \text{slice}(\alpha) \text{ for all } \alpha \text{ such that } 0 < \alpha < b(n-1), \text{ and}$$

$$\tau(\text{slice}(\overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1} \frown b(n-1) \frown \alpha)) = \text{slice}(b(n-1) + \alpha) \text{ for all } \alpha \leq \mathbf{e}.$$

On the other hand, if  $b(n-1) = 0$ , we complete the definition of  $\tau$  by setting

$$\tau(\text{slice}(\overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1} \frown 0 \frown 0)) = \text{slice}(0 \frown \overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1}), \text{ and}$$

$$\tau(\text{slice}(\overbrace{\mathbf{e} \dots \mathbf{e}}^{n-1} \frown 0 \frown \alpha)) = \text{slice}(\alpha) \text{ for all } \alpha \leq \mathbf{e}.$$

Condition (1) holds for  $\tau$  because once again we are building our map from the maps created in Proposition 4.3. To check condition (2), note that if  $b' \in \text{slice}(b \upharpoonright ht(b))$  and  $lh(b')$  is finite, then  $b' \in \text{slice}(\mathbf{eee} \dots \mathbf{e}b(n-1)\beta)$  for some  $\beta$ . If  $b(n-1) \neq 0$ , then  $\tau(b') \in \text{slice}(\mathbf{eee} \dots (b(n-1) + \beta))$  and thus  $lh(\tau(b')) = lh(b') - 1$ . If  $b(n-1) = 0$ , then  $\tau(b')$  has one less 0 as an entry and again,  $lh(\tau(b')) < lh(b')$ . Condition (3) and the fact that  $\tau$  is an order-preserving bijection are routine to prove and left to the reader.

We now look at the case where  $lh(b) = n$ , assuming we have proven the cases for smaller values of  $lh(b)$ . So  $b = b(0)b(1) \dots b(n-1)\mathbf{eee} \dots$ . Let  $k$  be the first entry of  $b$  not equal to  $\mathbf{e}$ . So  $b < \mathbf{e} \dots \mathbf{e}b(k)\mathbf{eee} \dots$  which we will call  $b'$ . By our induction hypothesis,  $A = \{l \in \kappa_e^\omega(X) \mid l \leq b'\} \simeq \kappa_e^\omega(X)$  via an isomorphism  $\tau'$ . Since  $b \in \text{slice}(b' \upharpoonright ht(b'))$  and  $lh(b)$  is finite, we know that  $lh(\tau'(b)) < lh(b)$ . Using our induction hypothesis again,  $A' = \{l \in \kappa_e^\omega(X) \mid l \leq \tau'(b)\} \simeq \kappa_e^\omega(X)$  via an isomorphism  $\tau''$ . Our candidate for  $\tau$  is thus  $\tau'' \circ \tau' \upharpoonright \{l \in \kappa_e^\omega(X) \mid l \leq b\}$ . It clearly satisfies condition (1).

We check condition (2). Suppose  $\hat{b} \in \text{slice}(b \upharpoonright ht(b))$  and  $lh(\hat{b})$  is finite. We know  $\hat{b} \in \text{slice}(b' \upharpoonright ht(b'))$  so  $lh(\hat{b}) > lh(\tau'(\hat{b}))$ . By condition (3) on  $\tau'$  (replacing  $b, b', b''$  with  $b', b, \hat{b}$ ), we know that  $\tau'(\hat{b}) \in \text{slice}(\tau'(b) \upharpoonright ht(\tau'(b)))$ . So when we apply the map  $\tau''$ , by condition (2) we know  $lh(\tau''(\hat{b})) > lh(\tau'' \circ \tau'(\hat{b}))$ . So  $lh(\hat{b}) > lh(\tau'' \circ \tau'(\hat{b}))$  and condition (2) is satisfied.

We check condition (3). Suppose  $b_1, b_2 \in \text{slice}(b \upharpoonright ht(b))$  where both  $ht(b_1)$  and  $ht(b_2)$  are finite and  $b_2 \in \text{slice}(b_1 \upharpoonright ht(b_1))$ . So  $b \in \text{slice}(b' \upharpoonright ht(b'))$  and  $b_1 \in \text{slice}(b \upharpoonright ht(b))$ , thus by our induction hypothesis

$$(1) \quad \tau'(b_1) \in \text{slice}(\tau'(b) \upharpoonright ht(\tau'(b)))$$

Also,  $b_1 \in \text{slice}(b' \upharpoonright ht(b'))$  and  $b_2 \in \text{slice}(b_1 \upharpoonright ht(b_1))$  so by induction

$$(2) \quad \tau'(b_2) \in \text{slice}(\tau'(b_1) \upharpoonright ht(\tau'(b_1)))$$

Thus using the induction hypothesis with equations 1 and 2, we deduce

$$\tau'' \circ \tau'(b_2) \in \text{slice}(\tau'' \circ \tau'(b_1) \upharpoonright ht(\tau'' \circ \tau'(b_1)))$$

thus satisfying condition (3).  $\square$

**Proposition 4.8.** *Given  $a, b \in \kappa_e^\omega - \{\mathbf{eee} \dots\}$ , the interval  $\{l \in \kappa_e^\omega(X) \mid a < l \leq b\}$  is isomorphic to  $\kappa_e^\omega(X)$  via a relatively height preserving isomorphism  $\tau$ .*

*Proof.* By the previous proposition, there exists an isomorphism  $\tau : \{l \in \kappa_e^\omega(X) \mid l \leq b\} \rightarrow \kappa_e^\omega(X)$ . By proposition 4.6, there exists an isomorphism  $\sigma : \{l \in \kappa_e^\omega(X) \mid \tau(a) < l\} \rightarrow \kappa_e^\omega(X)$ . (We are using the fact here that  $\tau$  is a relatively height preserving map and thus  $ht(\tau(a))$  is finite.) The map  $\sigma\tau$  restricted to the interval  $(a, b]$  is our desired relatively height preserving isomorphism.  $\square$

We now have enough machinery to ensure we can map any finite, strictly increasing sequence in  $\kappa_e^\omega(X)$  to any other such sequence via an automorphism of  $\kappa_e^\omega(X)$ , as long as both sequences consist of elements of finite height not equal to  $\mathbf{eee} \dots$ . (We simply use the above propositions to ensure isomorphisms of the spaces between the elements.) However, we have no such machinery for elements of infinite height. To produce such machinery, we will approximate the elements of infinite height with elements of finite height.

**Proposition 4.9.** *Given  $a \in \kappa_e^\omega(X)$ , if  $a \neq \mathbf{eee}$ , then there exist sequences  $\{r_i\}_{i < \omega}$ ,  $\{s_i\}_{i < \omega}$  of elements of  $\kappa_e^\omega$  such that*

1.  $r_0 < r_1 < \dots < r_i < \dots < a$  and  $\sup_i r_i = a$ .
2.  $s_0 > s_1 > \dots > s_i > \dots > a$  and  $\inf_i s_i = a$ .

*Proof.* The proof breaks into two parts, one for elements of finite height and one for elements of infinite height.

Suppose  $ht(a) = n$ . So  $a$  has the form  $a(0)a(1) \dots a(n-1)\mathbf{eee} \dots$ . We let

$$\begin{aligned} r_0 &= a(0)a(1) \dots a(n-1)0\mathbf{eee} \dots \\ r_1 &= a(0)a(1) \dots a(n-1)\mathbf{e}0\mathbf{eee} \dots \\ &\vdots \end{aligned}$$

$$r_k = a(0)a(1) \dots a(n-1) \overbrace{\mathbf{e} \dots \mathbf{e}}^k 0\mathbf{eee} \dots$$

and we let

$$\begin{aligned} s_0 &= a(0)a(1) \dots (a(n-1) + 1)0\mathbf{eee} \dots \\ s_1 &= a(0)a(1) \dots (a(n-1) + 1)00\mathbf{eee} \dots \\ &\vdots \end{aligned}$$

$$s_k = a(0)a(1) \dots (a(n-1) + 1) \overbrace{0 \dots 0}^{k+1} \mathbf{eee} \dots$$

It is clear that our definition of  $r_i$  suffices. The case for  $s_i$  is slightly more difficult. Suppose that  $a' \in \kappa_e^\omega(X)$  and  $a' > a$  but  $a' \leq s^i$  for all  $i$ . Since  $a < a' \leq s_0$ , we have that  $a \upharpoonright (n-1) = a' \upharpoonright (n-1) = s_0 \upharpoonright (n-1)$  and  $a(n-1) \leq a'(n-1) \leq a(n-1) + 1$ . If  $a'(n-1) = a(n-1)$ , then for some  $j > n-1$ ,  $a \upharpoonright j = a' \upharpoonright j$  and  $a(j) < a'(j)$ . But since  $a(j) = \mathbf{e}$ , this is impossible. Thus we must have that  $a'(n-1) = a(n-1) + 1$ . Note that since  $a' \leq s_0$  and  $a' \upharpoonright n = s_0 \upharpoonright n$ , we must have that  $a'(n) \leq s_0(n) = 0$ , and thus  $a'(n) = 0$ . Repeating this argument with  $s_i$  for all  $i \in \omega$ , it is easy to check

that for all  $m \geq n$ ,  $a'(m) = 0$ . But then  $a' \notin \kappa_e^\omega(X)$  as we supposed. Therefore  $\inf_i s_i = a$ .

Now suppose  $ht(a) = \omega$ . Thus  $a \in prop(X)$  and so there exists  $n$  such that the sequence  $\{a(m)\}_{m \geq n}$  is strictly increasing with all elements not equal to zero. Define  $r_i = (a \upharpoonright (n+i)) \smallfrown 0\mathbf{e}\mathbf{e}\mathbf{e}\dots$ . Define  $s_i = (a \upharpoonright (n+i)) \smallfrown \mathbf{e}\mathbf{e}\mathbf{e}\dots$ . It is easy to check that both of these definitions fit the conditions of the proposition.  $\square$

Up to now, to carry through some of our results, we have needed relatively height preserving automorphisms. At this point, we want to produce automorphisms mapping any element of  $\kappa_e^\omega(X)$  to any other element, regardless of height.

**Proposition 4.10.** *For any  $n \in \omega$ , given any tuples  $\langle a_1, a_2, \dots, a_n \rangle, \langle b_1, b_2, \dots, b_n \rangle \in (\kappa_e^\omega(X))^n$  with  $a_1 < a_2 < \dots < a_n < \mathbf{e}\mathbf{e}\mathbf{e}\dots$  and  $b_1 < b_2 < \dots < b_n < \mathbf{e}\mathbf{e}\mathbf{e}\dots$ , then there exists  $\sigma \in Aut(\kappa_e^\omega(X))$  such that for all  $i \in \{1, \dots, n\}$ ,  $\sigma(a_i) = b_i$ .*

*Proof.* We prove the proposition via induction on  $n$ .

Suppose  $n = 1$ . We are trying to move  $a_1$  to  $b_1$ . Using the above proposition, we can find sequences  $\{r_i\}_{i < \omega}$  and  $\{s_i\}_{i < \omega}$  such that each sequence consists of elements of  $\kappa_e^\omega(X)$  of finite height,  $\{r_i\}_{i < \omega}$  ( $\{s_i\}_{i < \omega}$ ) is a strictly increasing (decreasing) sequences, and we have that  $\sup_i r_i = \inf_i s_i = a_1$ .

Denote by  $A$  an ordering consisting of one element. Let  $Y$  be the ordering obtained by concatenating the following orders

$$\{\omega \times \kappa_e^\omega(X)\} \cup A \cup \{\omega^* \times \kappa_e^\omega(X)\}$$

(where  $\omega^*$  is the ordering obtained by reversing the ordering on  $\omega$ .) Define the order-preserving isomorphism  $\sigma_a : \kappa_e^\omega(X) \rightarrow Y$  by the following:

$$\begin{aligned} \sigma_a(\{l : l \leq r^0\}) &= \{0\} \times \kappa_e^\omega(X) \text{ using proposition 4.7} \\ \sigma_a(\{l : r^i < l \leq r^{i+1}\}) &= (i+1) \times \kappa_e^\omega(X) \text{ for } i \in \omega \text{ by proposition 4.8} \\ \sigma_a(a^1) &= A \\ \sigma_a(\{l : s^{i+1} < l \leq s^i\}) &= (i+1)^* \times \kappa_e^\omega(X) \text{ for } i \in \omega \text{ by proposition 4.8} \\ \sigma_a(\{l : s^0 < l\}) &= \{0^*\} \times \kappa_e^\omega(X) \text{ by proposition 4.6} \end{aligned}$$

But for  $b$  we can construct a similar map  $\sigma_b : \kappa_e^\omega(X) \rightarrow Y$ . The map  $\sigma_b^{-1}\sigma_a$  is our desired automorphism. This proves the case for  $n = 1$ .

Now suppose  $n = m + 1$  and we have proven the theorem for all smaller values of  $n$ . By Proposition 4.4, we can find  $\hat{a}, \hat{b} \in \kappa_e^\omega(X)$  such that  $\hat{a}$  and  $\hat{b}$  have finite height, and  $a_1 < \hat{a} < a_2$  and  $b_1 < \hat{b} < b_2$ .

The interval  $\{l : l > \hat{a}\}$  (and respectively  $\{l : l > \hat{b}\}$ ) is isomorphic to  $\kappa_e^\omega(X)$  via an isomorphism  $\tau_{\hat{a}}$  ( $\tau_{\hat{b}}$ ). By our induction hypothesis, there exists an automorphism  $\sigma_1$  of  $\kappa_e^\omega(X)$  such that  $\sigma_1(\tau_{\hat{a}}(a_i)) = \tau_{\hat{b}}(b_i)$  for  $i \in \{2, \dots, n\}$ . Thus the order preserving map  $\tau_{\hat{b}}^{-1}\sigma_1\tau_{\hat{a}} : \{l : l > \hat{a}\} \rightarrow \{l : l > \hat{b}\}$  maps  $a_i$  to  $b_i$  for  $i \in \{2, \dots, n\}$ .

Similarly, the intervals  $\{l : l \leq \hat{a}\}$  and  $\{l : l \leq \hat{b}\}$  are also both isomorphic to  $\kappa_e^\omega(X)$ . By an argument like the above, we get an order preserving map  $\sigma_2 : \{l : l \leq \hat{a}\} \rightarrow \{l : l \leq \hat{b}\}$  which sends  $a_1$  to  $b_1$ . Putting the two maps together gives us our desired automorphism.  $\square$

**Proposition 4.11.** *Given  $c, d \in \kappa_e^\omega(X)$  of finite height, given any  $n$  and any two  $n$ -tuples  $\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_n \rangle \in (\kappa_e^\omega(X))^n$  with  $c < a_1 < \dots < a_n < d$  and  $c < b_1 < \dots < b_n < d$ , there exists  $\sigma \in Aut(\kappa_e^\omega(X))$  such that  $\sigma(a_i) = b_i$  for  $i \in \{1, \dots, n\}$  and  $\sigma$  fixes the set  $\{l : l \leq c\} \cup \{l : l \geq d\}$  pointwise.*

*Proof.* By proposition 4.8, there exists an isomorphism  $\tau : \{l \in \kappa_e^\omega(X) : c < l \leq d\} \rightarrow \kappa_e^\omega(X)$ . By proposition 4.10, there exists  $\sigma' \in \text{Aut}(\kappa_e^\omega(X))$  such that  $\sigma'(\tau(a_i)) = \tau(b_i)$  for  $i \in \{1, \dots, n\}$ . Our desired  $\sigma$  is the map  $\tau^{-1}\sigma'\tau$  on the interval  $(c, d]$  together with the identity map on the set  $\kappa_e^\omega(X) - (c, d]$ . We leave it to the reader to check the details.  $\square$

## 5. THE MANY PERMUTATION GROUP RESULT

We now move to the major theorem of the paper—a permutation group version of Shelah’s many model result for unstable theories.

**Theorem 5.1.** *Given  $T$ , a countable theory, and  $\kappa$ , a regular uncountable cardinal, if  $T$  is unstable, then there exist  $2^\kappa$  many models of  $T$  of size  $\kappa$  which are pairwise non-permutation group isomorphic.*

*Proof.* Let  $T'$  be a Skolemized extension of  $T$  in the language  $L' \supseteq L$ . Suppose the formula  $\phi(\bar{x}, \bar{y})$  witnesses the instability of  $T$  (and thus  $T'$ ) as given by Definition 3.1. By Corollary 3.2, given any linear ordering  $\Delta$ , there exists a model  $M(\Delta)$  of  $T'$  with the properties outlined in the corollary.

Since  $\kappa$  is a regular uncountable cardinal, there exists a set  $S$  of  $2^\kappa$  many subsets of  $\kappa$  consisting of cofinality  $\omega$  ordinals, such that if  $X, Y \in S$  and  $X \neq Y$ , then either  $X - Y$  or  $Y - X$  is stationary. (This follows from Lemma 7.6 in Section 8 of [J] which guarantees  $\kappa$  many disjoint stationary subsets  $A_\alpha$  of  $\kappa$ , each consisting of cofinality  $\omega$  ordinals. Then  $S = \{\bigcup_{\alpha \in Z} A_\alpha : Z \subseteq \kappa\}$ .) Given  $X, Y \in S$  with  $X \neq Y$ , we will show that  $M(\kappa_e^\omega(X)) \upharpoonright L$  and  $M(\kappa_e^\omega(Y)) \upharpoonright L$  are not permutation group isomorphic. The collection  $\{M(\kappa_e^\omega(X)) \upharpoonright L\}_{X \in S}$  will be our desired set of  $2^\kappa$  many pairwise non-permutation group isomorphic models of  $T$ .

Given  $X, Y \in S$ , we assume without loss of generality that  $X - Y$  is stationary. To ease notation, we will denote  $M(\kappa_e^\omega(X))$  by  $M(X)$  and  $M(\alpha_e^\omega(X))$  by  $M_\alpha(X)$  and similarly with  $Y$ . We will assume  $f : M(X) \upharpoonright L \rightarrow M(Y) \upharpoonright L$  is a permutation group isomorphism and look for a contradiction. Note that since  $\|M_\alpha(X)\| < \kappa$  for all  $\alpha < \kappa$  and for  $\beta < \kappa$  limit,  $M_\beta(X) = \bigcup_{\alpha < \beta} M_\alpha(X)$ , it is easy to see that  $C = \{\alpha < \kappa : f \upharpoonright M_\alpha(X) \text{ is a bijection between } M_\alpha(X) \text{ and } M_\alpha(Y)\}$  is a cub set. Thus  $(X - Y) \cap C$  is stationary.

From now on, we let  $\alpha$  range over the elements of  $(X - Y) \cap C$ . Recall that we defined  $g_\alpha : \omega \rightarrow \alpha$  as an element of  $\text{prop}(X)$ , and thus the  $n$ -tuple indexed by  $g_\alpha$ , namely  $\bar{a}_{g_\alpha}$ , is in  $M(X)$ . Therefore  $f(\bar{a}_{g_\alpha}) \in M(Y)$  and has the form

$$f(\bar{a}_{g_\alpha}) = \langle t_1^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}), \dots, t_q^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}) \rangle$$

for some  $m, q \in \omega$ , some choice of  $q$  many  $m$ -ary terms  $t_i^\alpha \in L'$ , and some choice of elements  $l_1^\alpha < \dots < l_m^\alpha \in \kappa_e^\omega(Y)$ .

We wish to count the number of possible types of  $\langle t_1^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}), \dots, t_q^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}) \rangle$  over  $M_\alpha(Y)$  in  $M(Y)$ . To start, we will focus on cuts. Given  $l \in \kappa_e^\omega(Y)$ , we define the cut  $l$  induces in  $\alpha_e^\omega(Y)$  to be the ordered pair  $\{\{l' \leq l : l' \in \alpha_e^\omega(Y)\}, \{l' \geq l : l' \in \alpha_e^\omega(Y)\}\}$ .

By indiscernibility, given  $m, q \in \omega$ , and  $q$  many  $m$ -ary terms  $t^j \in L'$ , and given  $l_1 < \dots < l_m \in \kappa_e^\omega(Y)$  and  $l'_1 < \dots < l'_m \in \kappa_e^\omega(Y)$ , if for all  $i \in \{1, \dots, m\}$ ,  $l_i$  and  $l'_i$  induce the same cut in  $\alpha_e^\omega(Y)$ , then the type

$$tp_{M(Y)}(\langle t^1(\bar{a}_{l_1}, \dots, \bar{a}_{l_m}), \dots, t^q(\bar{a}_{l_1}, \dots, \bar{a}_{l_m}) \rangle; M_\alpha(Y))$$

is the same as

$$tp_{M(Y)}(\langle t^1(\bar{a}_{l'_1}, \dots, \bar{a}_{l'_m}), \dots, t^q(\bar{a}_{l'_1}, \dots, \bar{a}_{l'_m}) \rangle; M_\alpha(Y)).$$

To count the number of types, we first count the number of cuts in  $\alpha_e^\omega(Y)$  induced by members of  $\kappa_e^\omega(Y)$ . Clearly, the number of cuts induced in  $\alpha_e^\omega(Y)$  by members of  $\alpha_e^\omega(Y)$  equals  $\|\alpha_e^\omega(Y)\| = \|\alpha\| + \aleph_0$ .

If  $l \in \kappa_e^\omega(Y) - \alpha_e^\omega(Y)$ , there exists a least  $i$  such that  $l(i)$  is greater than or equal to  $\alpha$ , but not equal to  $\mathbf{e}$ . Having found this  $i$ , we make the following claim:

**Claim 1.** *The cut  $l$  induces in  $\alpha_e^\omega(Y)$  is the same as the element  $\hat{l} = (l \upharpoonright i) \frown \mathbf{e} \frown 000 \dots \in (\kappa + 1)^\omega$  induces in  $\alpha_e^\omega(Y)$ .*

*Proof.* Suppose  $l' \in \alpha_e^\omega(Y)$  and  $l' < l$ . Then there exists  $j$  such that  $l' \upharpoonright j = l \upharpoonright j$ , but  $l'(j) < l(j)$ . If  $j < i$ , then  $l' \upharpoonright j = l \upharpoonright j = \hat{l} \upharpoonright j$ . But since  $l \upharpoonright i = \hat{l} \upharpoonright i$ , we have that  $l'(j) < l(j) = \hat{l}(j)$  and thus  $l' < \hat{l}$ . If on the other hand,  $j \geq i$ , then  $l' \upharpoonright i = l \upharpoonright i = \hat{l} \upharpoonright i$ . Now since  $\alpha \leq l(i) < \mathbf{e}$ , but  $l' \in \alpha_e^\omega(Y)$ ,  $l'(i) < \alpha$ , so  $l'(i) < l(i) < \hat{l}(i) = \mathbf{e}$ . Thus  $l' < \hat{l}$ .

Suppose  $l' \in \alpha_e^\omega(Y)$  and  $l' > l$ . Then there exists  $j$  such that  $l \upharpoonright j = l' \upharpoonright j$  but  $l(j) < l'(j)$ . If  $j < i$ , then  $\hat{l} \upharpoonright j = l \upharpoonright j = l' \upharpoonright j$ , but  $\hat{l}(j) = l(j) < l'(j)$  so  $\hat{l} < l'$ . If  $j \geq i$ , then  $\hat{l} \upharpoonright i = l \upharpoonright i = l' \upharpoonright i$ . Since  $\alpha \leq l(i) < \mathbf{e}$  and  $l < l'$ , we must have that  $l'(i) = \mathbf{e}$ . Thus  $\hat{l}(i) = l'(i) = \mathbf{e}$  but for all  $k > i$ ,  $0 = \hat{l}(k) \leq l'(k)$ . Since  $l'$  eventually becomes constant with value  $\mathbf{e}$  or eventually becomes an increasing sequence,  $\hat{l} < l'$ .  $\square$

We have shown that the cut an element of  $\kappa_e^\omega(Y)$  induces in  $\alpha_e^\omega(Y)$  is the same as the cut induced by a member of  $\alpha_e^\omega(Y)$ , or the same as the cut induced by an element with initial segment consisting of entries either equal to  $\mathbf{e}$  or less than  $\alpha$  followed by the string  $000 \dots$ . Let  $D_\alpha$  be the set of all such elements. Then given any  $l \in \kappa_e^\omega(Y)$ , the cut  $l$  induces in  $\alpha_e^\omega(Y)$  is the same as a member of  $D_\alpha \cup \alpha_e^\omega(Y)$ . Therefore, the number of possible cuts of  $\alpha_e^\omega(Y)$  induced by a member of  $\kappa_e^\omega(Y)$  is equal to  $\|\alpha_e^\omega(Y)\| + \|D_\alpha\|$ , which can easily be seen to be less than or equal to  $\|\alpha\| + \aleph_0$ .

In summary, given  $\alpha \in (X - Y) \cap C$ , the type of  $f(\bar{a}_{g_\alpha})$  over  $M_\alpha(Y)$  in  $M(Y)$  is determined by the choice of terms  $t_1^\alpha, \dots, t_q^\alpha$ , their arity  $m$ , and the cuts in  $\alpha_e^\omega(Y)$  induced by  $l_1^\alpha, \dots, l_m^\alpha$ . Note that for different values of  $\alpha$ , the type is determined by cuts in different orderings.

Recall that  $(X - Y) \cap C$  is stationary. By using a variant of Fodor's Theorem along with the fact that we are working with a countable language, we can find a set of terms  $t_1, \dots, t_q \in L'$  for some  $q \in \omega$  and some stationary set  $A \subset (X - Y) \cap C$ , such that for all  $\alpha \in A$  and all  $j \in \{1, \dots, q\}$ ,  $t_j^\alpha = t_j$ . Assume that each  $t_j$  is an  $m$ -ary term. For each  $i \in \{1, \dots, m\}$  and for all  $\alpha \in A$ , the cut in  $\alpha_e^\omega(Y)$  induced by  $l_i^\alpha$  is the same cut induced by an element  $\hat{l}_i^\alpha \in \alpha_e^\omega(Y) \cup D_\alpha$ . Since for all  $\alpha < \kappa$ ,  $\|\alpha_e^\omega(Y) \cup D_\alpha\| < \kappa$ , for all limit ordinals  $\delta$ ,  $\delta_e^\omega(Y) \cup D_\delta = \bigcup_{\alpha < \delta} (\alpha_e^\omega(Y) \cup D_\alpha)$ , and for all  $\alpha < \beta < \kappa$ ,  $\alpha_e^\omega(Y) \cup D_\alpha \subseteq \beta_e^\omega(Y) \cup D_\beta$ , we can again use a well-known variant of Fodor's Theorem on the map  $\alpha \mapsto \hat{l}_i^\alpha$ . By applying the theorem to  $A$  and  $i = 1$ , we can find a stationary set  $A_1 \subseteq A$  and an element  $\hat{l}_1 \in \bigcup_{\alpha < \kappa} (\alpha_e^\omega(Y) \cup D_\alpha)$  such that for all  $\alpha \in A_1$ ,  $\hat{l}_1^\alpha = \hat{l}_1$ . By iterating the process, we can find a stationary set  $A' \subset (X - Y) \cap C$ , a set of terms  $t_1, \dots, t_q \in L'$ , and a set of elements  $\hat{l}_1, \dots, \hat{l}_m$ , such that for all  $\alpha \in A'$ , for all  $j \in 1, \dots, q$ ,  $t_j^\alpha = t_j$ , and for all  $i \in \{1, \dots, m\}$ ,  $\hat{l}_i^\alpha = \hat{l}_i$ .

Let  $\alpha < \beta$  be two ordinals in  $A'$ . By the above, since for all  $i$ ,  $l_i^\alpha$  and  $l_i^\beta$  induce the same cut in  $\alpha_e^\omega(Y)$  as does  $\hat{l}_i$ , we have that

$$\begin{aligned} & tp_{M(Y)}(\langle t_\alpha^1(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}), \dots, t_\alpha^q(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}) \rangle; M_\alpha(Y)) \\ &= tp_{M(Y)}(\langle t_\alpha^1(\bar{a}_{l_1^\beta}, \dots, \bar{a}_{l_m^\beta}), \dots, t_\alpha^q(\bar{a}_{l_1^\beta}, \dots, \bar{a}_{l_m^\beta}) \rangle; M_\alpha(Y)) \end{aligned}$$

We make a claim, which if true, proves the theorem:

**Claim 2.** *The tuples  $\langle t_1^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}), \dots, t_q^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}) \rangle$  and  $\langle t_1^\beta(\bar{a}_{l_1^\beta}, \dots, \bar{a}_{l_m^\beta}), \dots, t_q^\beta(\bar{a}_{l_1^\beta}, \dots, \bar{a}_{l_m^\beta}) \rangle$  are in the same orbit over  $M_\alpha(Y)$  in  $M(Y)$ .*

*Proof of Theorem 5.1 from Claim 2.* Since  $f$  is a permutation group isomorphism, if the above elements are in the same orbit over  $M_\alpha(Y)$  in  $M(Y)$ , then by mapping by  $f^{-1}$  and taking the reduct to the language  $L$ , we have

$$tp_{M(X) \upharpoonright L}(\bar{a}_{g_\alpha}; M_\alpha(X)) = tp_{M(X) \upharpoonright L}(\bar{a}_{g_\beta}; M_\alpha(X))$$

since  $\bar{a}_{g_\alpha}$  and  $\bar{a}_{g_\beta}$  are in the same orbit over  $M_\alpha(X)$  in  $M(X)$ . (Here we are using the fact that since  $\alpha, \beta \in (X - Y) \cap C$ ,  $f^{-1}(M_\alpha(Y)) = M_\alpha(X)$ .)

Since  $g_\beta$  is a strictly increasing sequence with limit  $\beta$ , there exists a least  $i \in \omega$  such that  $g_\beta(i) > \alpha$ . Let  $\gamma = g_\alpha(i) + 1$ . Since  $g_\alpha$  is a strictly increasing sequence with limit  $\alpha$ , where  $\alpha$  is a limit ordinal,  $\gamma < \alpha$ . Let  $c = (g_\alpha \upharpoonright i) \frown \gamma \frown \mathbf{eee} \in \alpha_e^\omega(X)$ . It is easy to check that  $g_\alpha < c < g_\beta$ . However, this implies that  $M(X) \models \phi(\bar{a}_{g_\alpha}, \bar{a}_c) \wedge \neg \phi(\bar{a}_{g_\beta}, \bar{a}_c)$  (where  $\phi(\bar{x}, \bar{y})$  is the formula witnessing the instability of  $T'$ .) Thus  $\bar{a}_{g_\alpha}$  and  $\bar{a}_{g_\beta}$  have different types over  $M_\alpha(X)$  in  $M(X)$ , contradicting the above equation. Therefore  $f$  cannot be a permutation group isomorphism, giving us the contradiction desired at the beginning of the proof.  $\square$

It is interesting to note that at this point, our proof has closely followed the standard proof of this theorem where the goal has been to produce non-isomorphic models as opposed to non-permutation group isomorphic models. At this point, however, since we are working with weaker tools, more work is required. (Note that Claim 2 is trivial to prove if  $f$  is an isomorphism and we want the tuples to have the same type instead of orbit.)

*Proof of Claim 2.* To show that the tuples  $\langle t_1^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}), \dots, t_q^\alpha(\bar{a}_{l_1^\alpha}, \dots, \bar{a}_{l_m^\alpha}) \rangle$  and  $\langle t_1^\beta(\bar{a}_{l_1^\beta}, \dots, \bar{a}_{l_m^\beta}), \dots, t_q^\beta(\bar{a}_{l_1^\beta}, \dots, \bar{a}_{l_m^\beta}) \rangle$  are in the same orbit over  $M_\alpha(Y)$  in  $M(Y)$ , it suffices by Corollary 3.2 to find a 1-1, onto, order preserving map  $\sigma : \kappa_e^\omega(Y) \rightarrow \kappa_e^\omega(Y)$  such that for all  $i \in \{1, \dots, m\}$ ,  $\sigma(l_i^\alpha) = l_i^\beta$  and  $\sigma$  fixes the set  $\alpha_e^\omega(Y)$  pointwise.

Recall that we have chosen our elements such that  $l_1^\alpha < \dots < l_m^\alpha$ . Also recall that for all  $i \in \{1, \dots, m\}$ , the elements  $l_i^\alpha$  and  $l_i^\beta$  induce the same cuts in  $\alpha_e^\omega(Y)$ . So if for some  $j$ ,  $l_j^\alpha \in \alpha_e^\omega(Y)$ , then  $l_j^\alpha = l_j^\beta$ . Therefore, if  $\sigma$  is as above, since it fixes  $\alpha_e^\omega(Y)$ , it will have the property that  $\sigma(l_j^\alpha) = l_j^\beta$ .

Therefore, without loss of generality, we will assume that the elements of the tuples  $l_1^\alpha, \dots, l_m^\alpha$  and  $l_1^\beta, \dots, l_m^\beta$  are all in  $\kappa_e^\omega(Y) - \alpha_e^\omega(Y)$ .

We can define an equivalence relation  $\sim$  on the set  $\{1, \dots, m\}$  by setting  $i \sim j$  if  $l_i^\alpha$  and  $l_j^\alpha$  induce the same cut in  $\alpha_e^\omega(Y)$ . (Note that this is the same equivalence relation defined by setting  $i \sim j$  if  $l_i^\beta$  and  $l_j^\beta$  induce the same cut in  $\alpha_e^\omega(Y)$ .) If  $i \sim k$  and  $i < j < k$ , since  $l_i^\alpha < l_j^\alpha < l_k^\alpha$ , we have that  $i \sim j \sim k$ . Therefore, the equivalence classes of  $\sim$  will be sets of consecutive numbers.

We make the following claim:

**Claim 3.** *Given  $E$ , an equivalence class of  $\sim$ , there exists  $\hat{l}_1, \hat{l}_2 \in \kappa_e^\omega$  such that  $\bigcup_{i \in E} \{l_i^\alpha\} \cup \bigcup_{i \in E} \{l_i^\beta\}$  is a subset of the interval  $(\hat{l}_1, \hat{l}_2)$  in  $\kappa_e^\omega(Y)$ , and  $(\hat{l}_1, \hat{l}_2) \cap \alpha_e^\omega(Y) = \emptyset$ .*

*Proof.* Suppose  $E = \{m_1, \dots, m_k\}$ . Define  $l' = \min(l_{m_1}^\alpha, l_{m_1}^\beta)$  and  $l'' = \max(l_{m_k}^\alpha, l_{m_k}^\beta)$ . Note that the interval  $[l', l'']$  contains no elements from  $\alpha_e^\omega(Y)$  (since otherwise, for some  $i, j, m_i \not\sim m_j$ .)

Since  $l' \notin \alpha_e^\omega(Y)$ , there exists a least  $i$  such that  $\alpha \leq l'(i) < \mathbf{e}$ . Find the least  $j > i$  such that  $l'(j) \neq 0$  and define  $\hat{l}_1 = (l' \upharpoonright j) \frown 0 \frown \mathbf{e}$ . Thus  $\hat{l}_1 < l'$ . Note that since  $l' \upharpoonright j = \hat{l}_1 \upharpoonright j$  and this finite sequence contains an entry greater than or equal to  $\alpha$  but not equal to  $\mathbf{e}$ , the interval  $[\hat{l}_1, l']$  contains no member of  $\alpha_e^\omega(Y)$ .

Similarly  $l'' \notin \alpha_e^\omega(Y)$ , so there exists a least  $i$  such that  $\alpha \leq l''(i) < \mathbf{e}$ . Let  $\gamma = l''(i) + 1$  and define  $\hat{l}_2 = (l'' \upharpoonright i) \frown \gamma \frown \mathbf{e}$ . Then  $l'' < \hat{l}_2$ , and as above one can check that no member of  $\alpha_e^\omega(Y)$  lies in the interval  $[l'', \hat{l}_2]$ .  $\square$

Given  $E$ , an equivalence class of  $\sim$ , let  $I_E$  be the interval defined above. Given two such equivalence classes,  $E_1$  and  $E_2$ , suppose that  $E_1 \neq E_2$ . Without loss of generality, we may assume that there exists  $j \in E_1, i \in E_2$  such that  $j < i$ . By our definition of  $\sim$ , there exists  $c \in \alpha_e^\omega(Y)$  such that  $l_j^\alpha < c < l_i^\alpha$ . Since  $I_{E_1} \subseteq \{l \in \kappa_e^\omega(Y) : l < c\}$  and  $I_{E_2} \subseteq \{l \in \kappa_e^\omega(Y) : l > c\}$ , we have that  $I_{E_1} \cap I_{E_2} = \emptyset$ .

By Proposition 4.11, for each interval  $I_E$  written as  $(\hat{l}_1, \hat{l}_2)$ , since both  $\hat{l}_1$  and  $\hat{l}_2$  have finite height, we can find a 1-1, onto, order-preserving map  $\tau_E : \kappa_e^\omega(Y) \rightarrow \kappa_e^\omega(Y)$  such that for all  $i \in E$ ,  $\tau_E(l_i^\alpha) = l_i^\beta$ , and  $\tau_E$  fixes the set  $\kappa_e^\omega(Y) - I_E$  pointwise (and thus fixes the set  $\alpha_e^\omega(Y)$  pointwise.) By constructing such a  $\tau_E$  for each of the finitely many equivalence classes of  $\sim$ , and then composing the maps, we obtain our desired 1-1, onto, order-preserving map  $\sigma : \kappa_e^\omega(Y) \rightarrow \kappa_e^\omega(Y)$  such that  $\sigma$  fixes the set  $\alpha_e^\omega(Y)$  pointwise, and that for all  $i \in \{1, \dots, m\}$ ,  $\sigma(l_i^\alpha) = l_i^\beta$ .  $\square$

Note that in our discussion, we have made great use of the fact that  $\kappa$  was regular. An open question is whether the requirement of regularity in Theorem 5.2 is necessary.

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