SOME RESULTS ON PERMUTATION GROUP ISOMORPHISM AND CATEGORICITY

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Abstract. We extend Morley's Theorem to show that if a theory is κ-p-categorical for some uncountable cardinal κ, it is uncountably categorical. We then discuss ω-p-categoriticy and provide examples to show that similar extensions for the Baldwin-Lachlan and Lachlan Theorems are not possible.

1. The basic definitions

We first start by defining a permutation group:

Definition 1.1. A permutation group is a pair $\langle X, G\rangle$ (where X is a set and G is a group) together with an action of G on X such that for any $\sigma \in G$, if for all $x \in X$, $\sigma(x) = x$, then σ is the identity function.

Definition 1.2. Given two permutation groups $\langle X_1, G_1\rangle$ and $\langle X_2, G_2\rangle$, we say that the two permutation groups are isomorphic if there exists a bijection $f : X_1 \to X_2$ such that the map $\sigma \mapsto f\sigma f^{-1}$ is an isomorphism of the groups G_1 and G_2.

Definition 1.3. Given a model M of a theory T, the permutation group associated with the model M is the pair $\langle |M|, Aut(M)\rangle$ (where $|M|$ stands for the universe of the model, and $Aut(M)$ stands for the group of automorphisms of M).

Definition 1.4. Given two models M and N, we will say that M is permutation group isomorphic to N if the permutation group $\langle |M|, Aut(M)\rangle$ is isomorphic to

1991 *Mathematics Subject Classification.* Primary 03C45.

$\langle |N|, Aut(N)\rangle$. Given a theory T, we will say T is κ-p-categorical if all of the models of T of size κ are permutation group isomorphic.

Our goal is to explore whether theorems about categoricity can be extended to p-categoricity. In particular, we will show:

Theorem 1.5. *Given T, a countable complete theory, if T is κ-p-isomorphic for some uncountable cardinal κ, then T is λ-categorical for all uncountable cardinals.*

We will also provide counterexamples for similar extensions of the Baldwin-Lachlan and Lachlan theorems.

We end this section with two general and needed facts about permutation group isomorphisms. The first has a trivial proof; the second is a simple generalization of the corresponding fact about types (see, for example, Lemma 0.9 in [P]).

Proposition 1.6. *Given a model M permutation group isomorphic to a model M' via a map f, if $\bar{a}$ and $\bar{b}$ are in the same orbit over A in M, then $f(\bar{a})$ and $f(\bar{b})$ are in the same orbit over $f(A)$ in M'.*

Proposition 1.7. *Given a model M and a cardinal λ, suppose that given any $B \subseteq M$ such that $||B|| \leq \lambda$, there are at most $\lambda + \aleph_0$ many 1-orbits over B in M. Then given any $n \in \omega$, and any set $B \subseteq M$ such that $||B|| \leq \lambda$, there are at most $\lambda + \aleph_0$ many n-orbits over B in M.*

2. A version of Morley's Theorem

To prove our version of Morley's Theorem, we will show T is ω-stable and uni-dimensional. From now, T will refer to a countable, complete theory which is κ-p-categorical for some uncountable cardinal κ.

We will begin by reviewing the ordering introduced in [S].Given an uncountable cardinal κ, we will define our ordering κ_e^ω to be the subset of $(\kappa+1)^\omega$ consisting of all sequences which eventually become constant with value κ. (For the sake of clarity, we will write κ as $\mathbf{e}$ to indicate that it is the last (or **end**) element.) We give κ_e^ω the usual lexicographical ordering.

We will refer to elements of κ_e^ω using the variables a, b, c, d, l and use the notation $a(i)$ to refer to the ith entry of a. We will often write an element $a \in \kappa_e^\omega$ in the form $a = a(0)a(1)a(2)\ldots$.To refer to n-tuples from κ_e^ω, we will use the notation $\langle a_1, \ldots, a_n \rangle$. We also make the following definition:

Definition 2.1. For $l \in \kappa_e^\omega$, let $ht(l)$ (the height of l) be the least value of n such that for all $m \geq n$, $l(m) = \mathbf{e}$.

The important fact needed about κ_e^ω is the following:

Proposition 2.2. *Given $c, d \in \kappa_e^\omega$, given any n and any two n-tuples $\langle a_1, \ldots, a_n \rangle, \langle b_1, \ldots, b_n \rangle \in (\kappa_e^\omega)^n$ with $c < a_1 < \cdots < a_n < d$ and $c < b_1 < \cdots < b_n < d$, there exists $\sigma \in Aut(\kappa_e^\omega)$ such that $\sigma(a_i) = b_i$ for $i \in \{1, \ldots, n\}$ and σ fixes the set $\{l : l \leq c\} \cup \{l : l \geq d\}$ pointwise.*

Proof. See Proposition 4.11 of [S]. □

We will let T_1 be the theory of a dense linear ordering with a distinguished right endpoint. (Our language will consist of the relation $<$ and the constant e.) Clearly, κ_e^ω is a model of T_1. Using a standard elimination of quantifiers argument, we can show that every formula $\varphi(v_0, \ldots v_n)$ is T_1-equivalent to a Boolean combination of formulas of the form $v_i < v_j$, $v_i = v_j$, and $v_i = e$ where $i, j \in \{0, \ldots, n\}$. Using this, we can show:

Theorem 2.3. *Given $\lambda < \kappa$ and $B \subset \kappa_e^\omega$ with $\|B\| = \lambda$, there are at most $\lambda + \aleph_0$ many 1-types over B realized in κ_e^ω.*

Proof. We first expand B. Let $B' = \{(b \restriction m) \frown \bar{d} \frown \mathbf{eee}\ldots : m, n \in \omega, \bar{d} \in \{0, \mathbf{e}\}^n, \text{ and } b \in B\}$. Note that $\mathbf{eee}\ldots \in B'$ and that $B \subseteq B'$ (since given any $b \in B$, $(b \restriction ht(b)) \frown \mathbf{eee}\ldots = b$.) Furthermore, $||B'|| = \aleph_0 + \lambda$, since for every $b \in B$ we are adding countably many elements to B' for each initial segment of b. Also, note that it is easy to show that if $b' \in B'$, then given any $m, n \in \omega$ and any $\bar{d} \in \{0, \mathbf{e}\}^n$, we have that $(b' \restriction m) \frown \bar{d} \frown \mathbf{eee}\ldots \in B'$. To prove the theorem, it will suffice to show that the number of 1-types over B' realized in κ_e^ω is less than or equal to $\aleph_0 + \lambda$.

Given $a \in \kappa_e^\omega$, we will define the cut a induces in B' to be the ordered pair $\langle \{b \in B' : b \leq a\}, \{b \in B' : b \geq a\} \rangle$. If $a, a' \in \kappa_e^\omega$ induce the same cut in B', then clearly both a and a' satisfy the same open formulas with parameters from B', so $tp_{\kappa_e^\omega}(a; B') = tp_{\kappa_e^\omega}(a'; B')$. Therefore, to prove the theorem, it suffices to show that the number of cuts induced in B' by elements of κ_e^ω is less than or equal to $\aleph_0 + \lambda$. Given $a \in \kappa_e^\omega$, we define the sequence $\{\hat{a}(n)\}_{n \in \omega}$ as follows:

(1) Let $\hat{a}(0)$ equal the least $\alpha \leq \kappa$ such that there exists $b \in B'$ with the property that $b(0) = \alpha$ and $b \geq a$. (Such an α exists since $\mathbf{eee}\ldots \in B'$.)

(2) Having defined $\hat{a}(0), \ldots, \hat{a}(n-1)$, we define $\hat{a}(n)$ to be the least $\alpha \leq \kappa$ such that there exists $b \in B'$ with the property that for all $i \in \{0, \ldots, n-1\}$, $b(i) = \hat{a}(i)$, $b(n) = \alpha$, and $b \geq a$. (Such a b exists since by induction $\hat{a}(0) \frown \cdots \frown \hat{a}(n-1)$ is an initial segment of some member of B', and thus by our construction of B', $\hat{a}(0) \frown \cdots \frown \hat{a}(n-1) \frown \mathbf{eee}\ldots \in B'$, and this element is greater than or equal to a.)

Let $\hat{a}$ be the sequence $\{\hat{a}(n)\}_{n\in\omega}$. Note that $\hat{a}$ is not necessarily in κ_e^ω, but is a member of $(\kappa+1)^\omega$.

Claim 1. *$a = \hat{a}$ iff $a \in B'$.*

Proof. If $a = \hat{a}$, by construction, $\hat{a} \upharpoonright ht(\hat{a})$ is the initial segment of some member of B'. By our construction of B', $\hat{a} = (\hat{a} \upharpoonright ht(\hat{a})) \frown \mathbf{eee}\ldots \in B'$. The other direction is left to the reader. □

Claim 2. *If $a < \hat{a}$, then $\hat{a}$ consists of an initial segment of some some member of B' followed by zeroes.*

Proof. If $a < \hat{a}$, find the least j such that $a(j) \neq \hat{a}(j)$. Thus $\hat{a} \upharpoonright j+1$ is an initial segment of some member of B', and thus $(\hat{a} \upharpoonright (j+1)) \frown \overbrace{0\ldots0}^{m} \frown \mathbf{eee}\ldots \in B'$ for all m. Since the latter element is greater than a for all m, $\hat{a}(k) = 0$ for all $k \geq j+1$. □

Claim 3. *For all $b \in B'$ and for all $a \in \kappa_e^\omega$, if $a \leq b$, then $\hat{a} \leq b$.*

Proof. If $a \in B'$, then $a = \hat{a}$, so the proof is obvious. If $a \notin B'$, we know $\hat{a} \notin B'$. By the above claim, there exists a greatest i such that $\hat{a} \upharpoonright i = b \upharpoonright i$. By our construction of $\hat{a}$, $\hat{a}(i) \leq b(i)$, but by our choice of i, $\hat{a}(i) < b(i)$. Thus $\hat{a} < b$. □

Claim 4. *Given $c, d \in \kappa_e^\omega$, then c and d induce the same cut in B' if and only if $\hat{c} = \hat{d}$.*

Proof. If c and d induce the same cut in B', then one can check by examining our construction of $\hat{c}$ and $\hat{d}$ that $\hat{c} = \hat{d}$. Suppose $\hat{c} = \hat{d}$, but c and d induce different cuts in B'. Without loss of generality, we may assume that $c < d$, $c, d \notin B'$,and

$b \in B'$ such that $c < b \leq d$. Now since $c \notin B'$, we have that $\hat{c} < c$. By arguments above, $\hat{c} < b$ and $d \leq \hat{d}$, so $\hat{c} \neq \hat{d}$. □

We need to show that the number of cuts induced in B' by an element of κ_e^ω is less than or equal to $\lambda + \aleph_0$. We have just shown that two elements c, d of κ_e^ω induce the same cut in B' if and only if $\hat{c} = \hat{d}$. Therefore we need to show that $||\{\hat{c} : c \in \kappa_e^\omega\}|| \leq \lambda + \aleph_0$. But if $c \in \kappa_e^\omega$, then either $c \in B'$ and thus $c = \hat{c}$, or $c \notin B'$ and thus $\hat{c}$ consists of an initial segment of some member of B' followed by the string $000\ldots$. Since there are at most $\lambda + \aleph_0$ many such $\hat{c}$, we have proven the theorem. □

Theorem 2.4. *Given $\lambda < \kappa$ and $B \subset \kappa_e^\omega$ with $||B|| = \lambda$, for all n, there are at most $\lambda + \aleph_0$ many n-orbits over B in κ_e^ω.*

Proof. By Theorem 1.7, it suffices to prove the theorem for $n = 1$. We will in fact show that given any 1-type $q \in S(B)$ realized in κ_e^ω, the set of realizations in κ_e^ω of q is the disjoint union of at most three 1-orbits over B in κ_e^ω. This, together with Theorem 2.3, will prove the theorem.

We can view κ_e^ω as a topological space in the obvious way: our basis of open sets will be all intervals of the form (c, d) or $(c, \mathbf{eee}\ldots]$ for $c, d \in \kappa_e^\omega$. Since κ_e^ω is a dense linear ordering, κ_e^ω is Hausdorff. Any $\sigma \in Aut(\kappa_e^\omega)$ will be a homeomorphism of the space.

Suppose $q \in S(B)$ is a 1-type realized in κ_e^ω. Given a, b realizing q, a and b must induce the same cut in B. Without loss of generality, we may assume $q \neq tp_{\kappa_e^\omega}(b, B)$ for any $b \in B$.

If $a_1 < a_2$ are realizations of q and limit points of B, then no other limit point realization of q lies between them. Given any limit point realization a of q, then for any $\sigma \in Aut_B(\kappa_e^\omega)$, $\sigma(a)$ is also a limit point realization of q. Thus the set of limit point realizations of q consists of at most two orbits.

If a, b are realizations of q such that $a < b$ and both elements are not limit points of B, then there exists an interval (d_1, d_2) containing a and b but no elements of B. By applying Theorem 2.2, we can find $\sigma \in Aut(\kappa_e^\omega)$ such that $\sigma(a) = b$ and σ fixes the sets $\{l \in \kappa_e^\omega : l \leq d_1\}$ and $\{l \in \kappa_e^\omega : l \geq d_2\}$. Thus a and b are in the same orbit over B in κ_e^ω. Thus the set of realizations of q is the union of at most three 1-orbits over B in κ_e^ω. □

The below proof follows closely the proof of the theorem when "orbits" is replaced by "types" in [C-K] (see Corollary 3.3.14.) To generalize the proof to orbits, we need to use information we proved about κ_e^ω.

Theorem 2.5. *Given $\kappa > \aleph_0$, there exists a model M of T such that $\|M\| = \kappa$ and, given any $A \subset M$ with $\|A\| < \kappa$, for all n, there are at most $\|A\| + \aleph_0$ many n-orbits over A in M.*

Proof. Expand T to a theory T' which has Skolem functions in the expanded language $L' \supseteq L$. Let M' be the Skolem hull of κ_e^ω. Our candidate for M is $M' \upharpoonright L$.

Given $A \subset M$ with $\|A\| = \lambda < \kappa$, we look at A as a subset of M' and write each element of A in the form $t(\bar{a})$, where $t(\bar{x})$ is a term in L' and $\bar{a} \in (\kappa_e^\omega)^{<\omega}$. Let Y be the set of all $\bar{a}$ used in the above process. Let B be the set $\{b \in \kappa_e^\omega : \exists \bar{a} \in Y\ \exists i\ (\bar{a})_i = b\}$. Note that $\|B\| \leq \lambda + \aleph_0$. Note also that if $\bar{c}, \bar{c}'$ are in the same orbit over B in M' then $\bar{c}, \bar{c}'$ are in the same orbit over A in M. To prove the theorem, it suffices to show that there are at most $\lambda + \aleph_0$ many n-orbits over B in M' for any n.

However, to show this fact, it suffices to show the following: given $B' \subset \kappa_e^\omega$ with $\|B'\| \leq \lambda+\aleph_0$, for all m, there are at most $\lambda+\aleph_0$ many m-orbits over B' in κ_e^ω. Call this statement (†). Suppose (†) is true. Given any finite set of terms $t_1, \ldots, t_n \in L'$, if $\bar{d}, \bar{d}'$ are in the same orbit over B when viewed as a subset of κ_e^ω, then the tuples $\langle t_1(\bar{d}), \ldots, t_n(\bar{d})\rangle$ and $\langle t_1(\bar{d}'), \ldots, t_n(\bar{d}')\rangle$ are in the same n-orbit over B when viewed as a subset of M'. Since by (†), there are at most $\lambda + \aleph_0$ many m-orbits over B in κ_e^ω for all $m \in \omega$, the collection of n-tuples $\{\langle t_1(\bar{d}), \ldots, t_n(\bar{d})\rangle : \bar{d} \in (\kappa_e^\omega)^{<\omega}\}$ has representatives from at most $\lambda + \aleph_0$ many n-orbits over B in M'. However, every member of $(M')^n$ has the form $\langle t_1(\bar{d}), \ldots, t_n(\bar{d})\rangle$ for some $\bar{d} \in (\kappa_e^\omega)^{<\omega}$ and some set of terms $t_1, \ldots, t_n \in L'$. Since L' is countable, there are only countably many choices for the terms $t_1, \ldots, t_n$. Thus given (†), there are at most $\lambda + \aleph_0$ many n-orbits over B in M' for any n.

It simply remains to state that (†) is a consequence of Theorem 2.4. □

Theorem 2.6. *If T is κ-p-categorical for some uncountable cardinal κ, then T is ω-stable.*

Proof. Suppose T is not ω-stable. So there exists a countable set A and $n \in \omega$ such that $\|S^n(A)\| > \aleph_0$. Let X be a set of size κ consisting of elements realizing at least $\aleph_1$ many different types in $S^n(A)$. Let N be a model of T of cardinality κ such that $A \cup X \subset N$. Then N has at least $\aleph_1$ many n-orbits over A. But since T is κ-p-categorical, N is permutation group isomorphic to the model M of size κ constructed in Theorem 2.5 via some isomorphism f. Since M has at most countably many n-orbits over $f(A)$, we have our contradiction. □

Theorem 2.7. *If T is κ-p-categorical for some uncountable cardinal κ, then T is uncountably categorical.*

Proof. Since T is ω-stable, T has a saturated, homogeneous model M of cardinality κ. Thus, given any set $A \subset M$ such that $||A|| < \kappa$, any orbit over A is either finite or of size κ. Since T is κ-p-categorical, this is true of all models of size κ.

Assume T is not unidimensional. Without loss of generality, there exists a countable model M_0 and two strongly regular types $p, q \in S(M_0)$ such that $p \perp q$ and p has Morley rank 1. Let I be an independent sequence of realizations of p of length ω; let J be an independent sequence of realizations of q of length κ. Let N be prime over $M_0 \cup I \cup J$.

Our goal is to show that there are only countably many realizations of p in N. Since the elements in I form an orbit of size at least $\aleph_0$, we would then have an infinite orbit of size less than κ. Note that if a is a realization of p, $a \not\downarrow_{M_0} I \cup J$, and thus by orthogonality, $a \not\downarrow_{M_0} I$. Since p had Morley rank 1, a satisfies an algebraic formula with parameters from $M_0 \cup I$. Since there are only countably many such formulas, there are only countably many realizations of p in N. □

3. ω-P-CATEGORICITY

Proposition 3.1. *Suppose T is uncountably categorical. Then T is ω-categorical iff T is ω-p-categorical.*

Proof. Note that the countably saturated countable model M of T has two properties preserved by p-isomorphism:

(1) There is some infinite orbit under $Aut(M)$.

(2) Given any finite tuple $\bar{a}$ in M, any infinite orbit O under $Aut_{\bar{a}}(M)$, and any finite tuple $\bar{b}$ extending $\bar{a}$, there exists an infinite subset $O' \subseteq O$ which is an orbit under $Aut_{\bar{b}}(M)$.

Suppose these conditions are true for some countable model N. Pick a finite tuple $\bar{a}$ and an infinite orbit O such that the type p of an element in O over $\bar{a}$ has minimum Morley rank. By using property 2) above and extending $\bar{a}$, we may assume this type is stationary. By using property 2) repeatedly, we obtain an infinite Morley sequence realizing p. By the consequences of the Baldwin-Lachlan theorem, N must be countably saturated. □

Note that the assumption of uncountable categoricity was needed: consider the theory T with disjoint unary predicates P_{ij} (for $i < \aleph_0$ and $j \leq \aleph_0$) such that P_{ij} contains exactly j elements if j is finite and contains infinitely many elements if $j = \aleph_0$. T is not ω-categorical, since we build models with k realizations of the unique nonisolated type q for all $k \leq \aleph_0$. But T is ω-p-categorical, since we can map any model M (with k realizations of q) to the prime model N by mapping q^M to P^N_{0k} and mapping P^M_{ik} to $P^N_{(i+1)k}$.

As a counterexample to a "p-categorical" version of the Baldwin-Lachlan Theorem (if T is κ-p-categorical for some uncountable cardinal κ, then T has either 1 or $\aleph_0$ many countable models), consider the theory T with disjoint unary predicates P_{ij} (for $i, j < \omega$) such that P_{ij} contains exactly j elements. Clearly, T is uncountably categorical, but as above, we can map models containing finitely many realizations of the nonisolated type q to the prime model as above. Thus, there are two models up to p-isomorphism, one with no realizations of q and one with countably many realizations of q.

Finally, s a counterexample to a "p-categorical" version of Lachlan's Theorem (if T has only finitely many models in an uncountable power up to p-isomorphism, then T is either countably or uncountably categorical), consider the theory T as above with the addition of a disjoint unary predicate Q containing infinitely many elements. Up to p-isomorphism, there are two models of size $\aleph_1$, one in which both the nonisolated type and the type defined by $Q(x)$ have $\aleph_1$ many realizations and the other where one of the two types has only countably many realizations.

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