Writing ConcepTests for a Multivariable Calculus Class

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Abstract: In a multivariable calculus course, students must master a large number of concepts in order to successfully learn the material. This paper will discuss one way of addressing this difficulty through the use of ConcepTests, that is, multiple choice questions given in the lecture that test understanding as opposed to calculation. In particular, we will look at various types of ConcepTests and the material they can cover.

Keywords: multivariable calculus, ConcepTests, small group work

Introduction

As I started preparing for Centenary's multivariable calculus course in fall of 2000, I was considering how I could help student understanding. I had taught the course the past two years and had incorporated MATLAB programs to help the students visualize surfaces, curves, and vector fields. Even with this help, a significant number of students had difficulties understanding multivariable and vector concepts, particularly in the latter half of the course. Unfortunately, due to requirements on coverage of course material (the syllabus covers almost all of the Harvard multivariable calculus book), I was not able to dramatically slow down the pace. My goal was to find a way to gauge and help improve student understanding and provide a solid foundation in the essential concepts.

A little background: our multivariable calculus class is offered every fall with between 15 and 20 students (Centenary has 860 undergraduates). These students include mathematics, physics, and engineering majors as well as a few students from other disciplines. The first time I taught the class, we met about 20% of the time in a computer lab equipped with MATLAB; the second year we met full-time in the lab. The class meets five days a week for 50 minutes a day with a typical week consisting of four days of lecture and one day to discuss homework. I saw students consistently having problems with the following concepts: dot product, interpretations of the gradient, Lagrange multipliers, integrating with cylindrical and spherical coordinates, and line and flux integrals. While students could sometimes perform calculations with the above, they often had troubles explaining the concepts or attacking unfamiliar problems with that material.

History of ConcepTests

ConcepTests were developed by Eric Mazur, a Harvard physics professor. He had noticed that his introductory physics students could handle computational problems, but could not solve similar problems if the calculations were removed and only the underlying concept assessed. In other words, students were mistaking 'plug and chug' problem solving skills as understanding.

To address this problem, Mazur started using ConcepTests - multiple choice questions that students could answer in their heads if they correctly understood the concepts. In lecture, a test would be presented, students would vote on the correct answer, break into small groups to discuss their votes, and then revote on the question. Students would therefore engage the concepts in class through their votes and discussions with their classmates. A typical lecture might consist of several tests, with less time spent on examples. A longer discussion of the implementation and effectiveness of ConcepTests can be found in [1]. In addition, Scott Pilzer in [2] shows how these tests can be implemented in a first-year calculus course.

Writing ConcepTests

In my fall 2000 multivariable calculus course, I used ConcepTests, with about 20 minutes of each class period devoted to the tests. Since my use of the test was similar to Scott Pilzer's, I would recommend looking at his article for suggestions on incorporating ConcepTests. Instead, I want to focus on the types of tests I wrote for my class and the material each type was best suited to. Roughly speaking, my tests fell into one of five types: visualization, comparison, translation, theorem-using, and theorem-provoking.

Visualization Tests: These tests were designed to help the students think three-dimensionally. One example is:

Example 1: The set of all points whose distance from the z-axis is 4 is the:

a) sphere of radius 4 centered on the z-axis

b) line parallel to the z-axis 4 units away from the origin

c) cylinder of radius 4 centered on the z-axis

d) plane $z = 4$

(As is typical for ConcepTests, note that this problem is answerable without symbolic calculation.) I used visualization tests early in the semester, when the class focused on the three-dimensional coordinate system and various cross-sections of surfaces. Later on in the semester, I used this type of test to help students visualize vector fields. In both cases, the tests were very useful in providing students with an intuition about these objects before we moved to graphing tools.

Comparison Tests: I used this type of test most often during the semester. The question involved was determining the sign of a quantity or its relative magnitude. One example is:

Example 2: In which direction is the directional derivative of $z = x^2 + y^2$ at the point (2,3) most positive? (We are using $\mathbf{i}$ and $\mathbf{j}$ as the unit vectors in the x and y directions.)

a) $\mathbf{i}$

b) $-\mathbf{i}-\mathbf{j}$

c) $-\mathbf{i}+\mathbf{j}$

d) $\mathbf{i}+\mathbf{j}$

(Note that I would use this test before giving the formula for computing directional derivatives with the dot product.) The benefits of a question like this is that it helps the students understand which parameters affect a certain quantity. I used these types of tests extensively when discussing vector arithmetic (including velocity and acceleration vectors and dot product), partial and directional derivatives, the gradient vector, and line and flux integrals.

Translation Tests: When I arrived at the chapters on integration, I wasn't sure how to use ConcepTests. Since we were using a CAS in class, I decided to focus less on calculation and more on *translating* a particular integral into a specific coordinate system. One example is:

Example 3: Which of the following is equivalent to $\int_{-5}^{5}\int_{0}^{3}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} x\,dy\,dz\,dx$?

a) $\int_0^{\pi}\int_0^3\int_0^3 r^2\cos(\theta)dzdrd\theta$

b) $\int_0^{\pi}\int_0^3\int_0^5 r^2\cos(\theta)dzdrd\theta$

c) $\int_0^{2\pi}\int_0^5\int_0^3 r\cos(\theta)dzdrd\theta$

d) $\int_0^{2\pi} \int_0^5 \int_0^3 r^2 \cos(\theta) dz dr d\theta$

Note that here the focus is on translating the rectangular coordinates into cylindrical and properly adjusting the integrand. Using tests like these, I was able to help students think through the setup of a multivariable integral. In addition to the sections on integration, I wrote ConcepTests like this for parametric curves and surfaces and the parametrization of line integrals.

Theorem-Using Tests: With these types of tests, I assessed whether the students knew how and when to apply a theorem. One example is:

Example 4: Which of the following facts about the vector field $\mathbf{F}(\mathbf{r}) = \mathbf{r}$ (where $\mathbf{r}$ is a vector starting from the origin) is implied by Stoke's Theorem?

a) The line integral from $(0, 0, 0)$ to $(1, 1, 1)$ is equal to $\frac{3}{2}$.

b) $\mathbf{F}(\mathbf{r}) = \mathbf{r}$ has positive divergence everywhere.

c) The line integral on any closed curve is zero.

d) The curl of $\mathbf{F}(\mathbf{r}) = \mathbf{r}$ is non-zero.

Note that I do not want the students to carry out a specific calculation using Stoke's Theorem, but to understand its consequences. I also used theorem-using tests extensively when we covered optimization and the classification of critical points. In both cases, I was able to help the students understand the power and limitations of the theorems we used.

Theorem-Provoking Tests: Occasionally throughout the course, I would use ConcepTests not as a way of assessing student understanding on covered material, but to ready them for new material. One example is:

Example 5: The plot in Figure 1 shows the gradient vectors for a (hidden) function $f(x, y)$ and a linear constraint. Which point is closest to the global min of $f(x, y)$ on this constraint?

a) A

b) B

c) C

d) D

My goal here was to help the students see that at a global minimum on a constraint, the gradients of the objective and constraint functions are parallel. I also used tests like this when we discussed normals to surfaces in preparation for parametrizing flux integrals. For those sections which were almost purely computational, this type of test gave me the opportunity to engage the students before we got to the symbolic manipulation.

I wrote a total of 89 ConcepTests for my multivariable calculus, covering almost all of the topics in the Harvard multivariable calculus book. (As of this writing, the tests were being placed online at Eric Mazur's Peer Instruction site at galileo.harvard.edu.)

Reaction

When I started using the ConcepTests I discovered how much they enhanced both the feedback in the class and my ability to address students' questions. The most valuable time for me was when I walked among the students as they were discussing a ConcepTest. I very quickly learned students' strengths and weaknesses, got a chance in class to discuss the math with them, and was able to encourage their ideas.

Two pieces of evidence at the end of the semester pointed to the effectiveness of the tests. First, I received some of the best written student evaluations in my career, with several students specifically stating how the ConcepTests had helped. Second, of the three times I have taught this course, this class was the most successful in keeping student interest. I had fewer students who stopped coming to class, stopped turning in homework, or had large drops in exam scores than in past semesters. When I taught the class in fall of 1998, I had 13 students initially enrolled, 12 who took the final, and 9 with a grade of C or above. In fall of 1999, there were 19 initially enrolled, 18 who took the final, and 14 with a C or above. In the fall of 2000 when I used the ConcepTests, there were 20 initially enrolled, 20 who took the final, and 18 with a C or above. These classes are not directly comparable - my exams in fall of 2000 did include ConcepTests while previous classes did not - but I did notice fewer students who 'gave up' throughout the semester.

Finally, one of the most interesting pieces of feedback I got was from a student who reported that when he started studying for one of my exams, the first thing he did was to go through all the ConcepTests. His reason behind this was not only to prepare for the ConcepTests on the exam, but because reviewing the tests helped him to go back in time to the day he learned the material. Apparently, the ConcepTests provided him with 'mental landmarks' in the course.

Future Plans

This first use was primarily intended as a proof of concept - would concept tests work in a mathematics classroom and would they enhance student understanding? The answer to the first question is definitively yes. The second question needs more exploration, but at the least, I saw increased student participation in class.

My future plans are to see how concept tests can be used in other classes. I'm planning on using them in a first semester calculus class and may use them in a numerical analysis course. I am also planning to gather more data about individual concept tests (including students' answers before and after discussion and their level of confidence).

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References

[1] Mazur, Eric. 1997. *Peer Instruction: A User's Manual.* New Jersey: Pentice-Hall.

[2] Pilzer, Scott. 2001. Peer Instruction in Physics and Mathematics. *Primus* 11(2): 185-192.

Biographical Sketch

Mark D. Schlatter received his Ph.D. in mathematics from the University of California at Berkeley. Originally specializing in mathematical logic with a focus on model theory, he has sinced branched out to nonnegative matrix theory, the mathematics of art, and curriculum development. After three years as a Visiting Assistant Professor at Truman State University in Kirksville, MO, he is now an Assistant Professor at Centenary College.

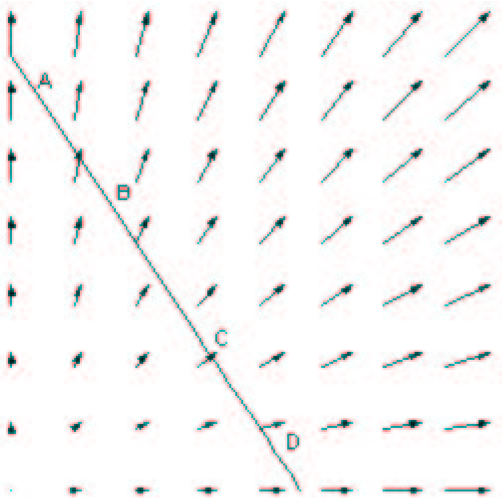

Figure 1: Plot for Example 5