Mirror Curves and Permutations

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Try tracing the designs in Figure 1 with your pencil, making sure not to change direction at any intersection. Done? You should have been able to trace the entire design without lifting your pencil in Figures 1a) and 1c), but you should have had to lift your pencil for Figure 1b).

Figure 1 goes here.

These three designs were all inspired by sona drawings — drawings made in the sand (usually in one smooth, continuous movement) by men of the Chokwe people. The Chokwe live in Angola, Zambia, and portions of Congo, and they use sona drawing in story-telling and in the initiation of young men. Paulus Gerdes, as part of his study of mathematics in Africa, has made a large study of sona, most recently in [1]. In that book, he states several facts about the number of curves needed to complete a sona drawing. We will review those facts and show how we can associate permutations with sona-inspired drawings.

To start with, it helps to understand how the above designs might be generated. Let us think of sona as mirror curves, that is, curves where

1. we start with a rectangular array of dots,
2. we surround the array with a box where the (horizontal or vertical) distance from the box to a point on the edge of the array equals half the (horizontal or vertical) distance between adjacent points,
3. we place 'mirrors' if need be in the grid, and
4. starting at the bottom of the box below a dot, we draw a line moving up and to the right at a 45 degree angle which bounces off any mirror or side of the box.

As examples, if you start at the 'S's in Figure 2 using the last instruction, you can recreate the designs in Figure 1 (after 'rounding off' where the curve hits a wall or mirror). Note that you need to start at two places and draw two curves to recreate Figure 1b).

Figure 2 goes here.

The first two drawings (with no mirrors inserted) are examples of what Gerdes calls plaited-mat designs (for fairly obvious reasons). The last drawing is an example of a lion's stomach design. In lion's stomach designs, there is always an odd number of columns and every other column, starting with the second, has horizontal mirrors between adjacent dots.

In [1], Gerdes states two facts about how many separate curves you need to draw to complete these sona-inspired drawings:

FACT 1 *A plaited-mat design on a m by n rectangular array requires* $gcd(m, n)$ *many curves.*

FACT 2 *A lion's stomach design on a m by n rectangular array requires* 1 *curve if* $n-1$ *is a multiple of* 4*; otherwise, it requires m.*

Let's investigate how we can prove these facts.

Plaited-Mat Designs

Gerdes discusses the proof of the first conjecture in [1]. We will cover it here to set the stage for lion's stomach designs. The first observation is that n curves are required for a plaited-mat design on an n by n array. An easy way to see this is to draw the box for the mirror curve and introduce a coordinate system where the bottom left-hand corner of the box is the origin and the space between the wall and a dot is one unit. As seen in Figure 3, a curve that starts at $(a, 0)$ hits the box at $(n+1, n+1-a)$, then at $(n+1-a, n+1)$ and $(0, a)$ before returning to its starting point.

Figure 3 goes here.

Besides seeing that n curves are necessary for the n by n square, it is important to note that a curve leaving the side of the square next returns to that same side at the same spot it left. Using this, we can now use

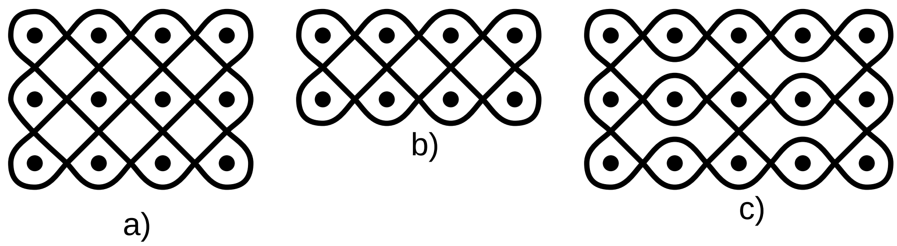

Figure 1: Three Examples of Sona Drawings

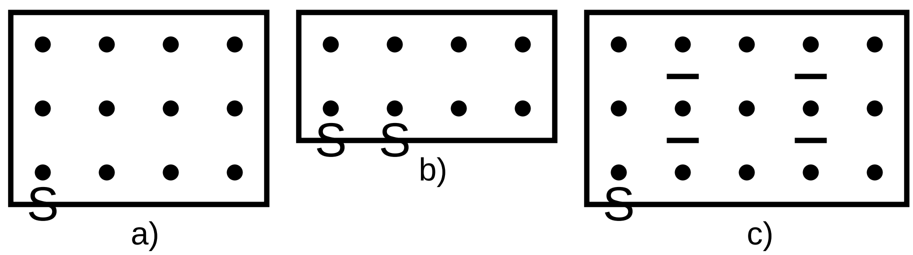

Figure 2: Sona as Mirror Curves

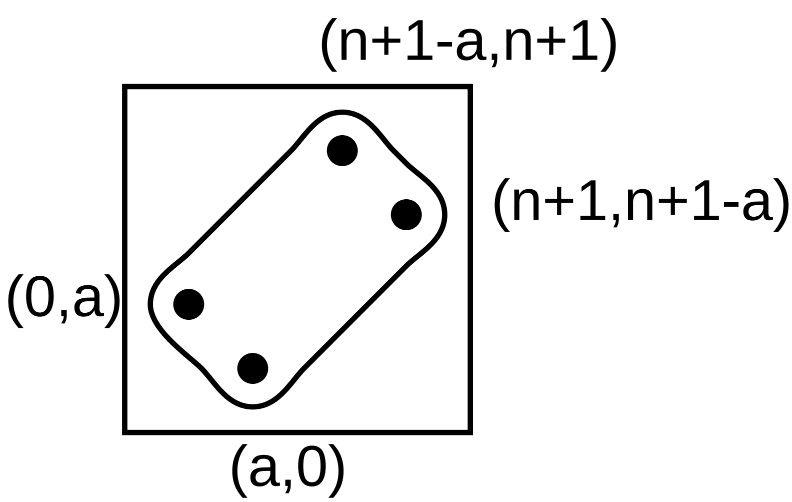

Figure 3: Plaited-mat design in an n by n array

a "cutting off squares" algorithm to find the number of curves needed to complete a plaited-mat design. We'll use the designs in Figures 1a) and 1b) to illustrate the algorithm.

In Figure 4a), we have taken our first design and cut off a 3 by 3 square. Note that the 3 by 1 array left over needs only one curve for a plaited-mat design and we can extend that curve to cover the entire 3 by 4 array by following the dotted lines when they leave the solid lines. In Figure 4b), we have followed the algorithm by cutting off a 2 by 2 square, leaving a 2 by 2 square that requires two curves to complete the design.

Figure 4 goes here.

In other words, if $f(m,n)$ is the number of curves needed to complete a plaited-mat design on an m by n array, we see that $f(n,n) = n$ and that $f(m,n) = f(m,n-m)$ (where $m < n$). (Note that the second fact is analogous to the subtraction algorithm for finding the greatest common divisor.) Add to these facts the easily checked results that $f(m,n) = f(n,m)$ and $f(m,1) = 1$ for any m, and a simple proof by induction yields Fact 1.

Lion's Stomach and Generalizations

Let us consider generalizing the lion's stomach design: instead of requiring every other column to have horizontal mirrors between adjacent dots, for each column we will decide either to have no mirrors (as in plaited-mat) or to have the horizontal mirrors between all dots (as in lion's stomach).

To start our analysis, Figure 5 shows the left-hand edge of a design with m rows. Note that we have numbered the strands after they cross and given paired strands different shadings to help distinguish them. We then show the new positions of these strands after a column with no mirrors. If we consider the column to be acting as a permutation on the strands, we can write this permutation as $\sigma = (2\ 4\ 6 \ldots 2m-2\ 2m\ 2m-1 \ldots 3\ 1)$. In other words, strand 2 has moved to the position first held by strand 4, strand 4 has moved to the position first held by strand 6, and so on. In the same figure, we show the new positions of the strands after a column with horizontal mirrors. We obtain the permutation $\tau = (1\ 2)(3\ 4)\dots(2m-1\ 2m)$.

Figure 5 goes here.

Given any generalized lion's stomach design on an m by n array, we can associate a word $x_1 x_2 \dots x_{n-2}$ where each x_i is either σ or τ. Note that we only need $n-2$ symbols since we are ignoring (for now) the columns on the sides.

We will reduce the words using multiplication as composition on the right. It is easy to check that σ^{2m} and τ^2 are the identity permutation, and with a little bit of work, you can also check that $\tau\sigma = \sigma^{2m-1}\tau$. Given the word associated with any design constructed as above, we can therefore reduce it to the form $\sigma^j \tau^k$ where $k = 0$ or $k = 1$ and $j \geq 0$. Moreover, the original design and the new design associated with the reduced word place the numbered strands in the same position as they reach the last column, so both drawings will require the same number of curves. We will explore the possible cases assuming our design uses an m by n array.

Case 1: The reduced word is τ. A quick look at Figure 6a) shows that the design will require as many curves as rows.

Figure 6 goes here.

Case 2: The reduced word has the form σ^j. In other words, the original design places numbered strands in the same positions before the last column as a plaited-mat design on an m by $j+2$ array. (The additional two columns are the ones on the ends.) The number of curves required will be $gcd(m, j{+}2)$.

Case 3: The reduced word has the form $\sigma^j\tau$. Figure 6b) shows the last three columns of the design associated with the reduced word. Note that just like in the plaited-mat discussion, curves that start at the left-hand side return to the left-hand side at the same point. Thus we can cut off the last three columns to produce a design with $j-2$ interior columns that requires the same number of curves. The number of curves required is $gcd(m, j)$. (Once again, we have added in the two columns on the ends.)

Note that we now have an easy proof of Fact 2. Lion's stomach designs have the associated words $\tau\sigma\tau$, $\tau\sigma\tau\sigma\tau, \dots$ or $\tau(\sigma\tau)^k$ for some value of k.

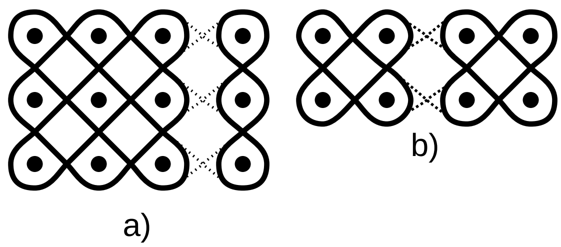

Figure 4: The “Cutting Squares” Algorithm

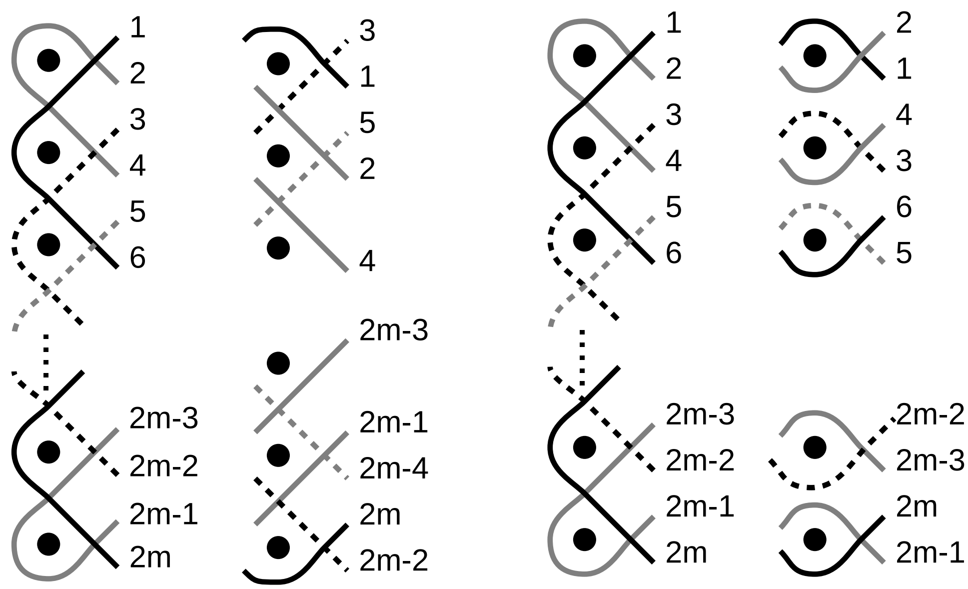

Figure 5: Moving Strands

Since $\tau\sigma\tau = \sigma^{2m-1}$, it is easy to check that $\tau(\sigma\tau)^k$ equals τ if k is even and equals σ^{2m-1} if k is odd. Remembering that a lion's stomach design always has an odd number of columns, a design with $4j+3$ columns will have the associated word $\tau(\sigma\tau)^{2j}$ and the reduced word τ and thus require as many curves as rows. A lion' stomach design with $4j+1$ columns will have the associated word $\tau(\sigma\tau)^{2j-1}$ and the reduced word σ^{2m-1} and, since $gcd(2m+1, m) = 1$, requires only one curve.

Questions

Clearly we can look at even more generalizations of the above designs. (For example, we could have horizontal mirrors between every other row instead of between every row.) The question arises: can we carry out a similar analysis? In particular, what type of reductions can we make and how do the reduced words tell us the number of curves needed?

Paulus Gerdes in [1] shows one more type of design that raises even more questions. Figure 7 shows a mirror curve (and associated box) which Gerdes refers to as 'chased chicken'. (Trace the curve and you'll appreciate the name!) Note that vertical mirrors are used to create this design resulting in fewer strands leaving the first column. Gerdes has proven that an m by n chased chicken design requires $gcd\left(\frac{m+1}{2}, \frac{n+2}{2}\right)$ curves. For the lion's stomach design, we used the 'base case' of the plaited-mat design. What 'base case' is needed for the chased chicken design and how can permutations be associated with this type of design or any design with vertical columns?

Figure 7 goes here.

References

[1] P. Gerdes, *Geometry From Africa: Mathematical and Educational Explorations*, Mathematical Association of America, 1999.

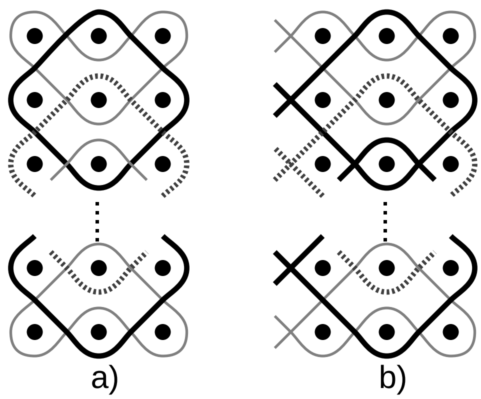

Figure 6: Cases 1 and 3

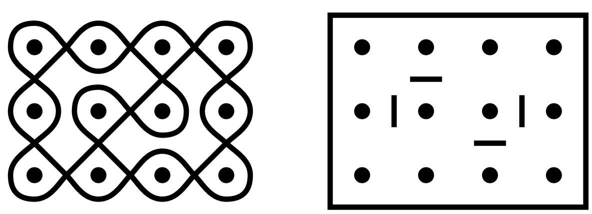

Figure 7: The Chased Chicken Design